



ON THE SPACE OF RIEMANNIAN METRICS

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ABSTRACT

The space of Riemannian metrics of a smooth compact manifold is investigated, particularly the natural action of the group of diffeomorphisms on that space. Both the space and the group are enlarged slightly, so that they can be endowed with differentiable structures. The orbits of the group are studied and a "slice" in the space of metrics transversal to these orbits is constructed. Also, the isotropy groups of the action are considered --- they are trivial for an open dense subset of the space of metrics.

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ON THE SPACE OF RIEMANNIAN METRICS

I. Introduction and Outline of Arguments.

Let M be a compact differentiable manifold, \mathcal{D} the group of diffeomorphisms of M , and \mathcal{M} the set of smooth Riemannian metrics on M . \mathcal{D} acts on \mathcal{M} from the right in a natural way; i.e., there exists a natural map $A: \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$ such that

$$A(\eta, A(\xi, \gamma)) = A(\xi\eta, \gamma) \quad \eta, \xi \in \mathcal{D}, \quad \gamma \in \mathcal{M}.$$

It is the purpose of the present work to investigate this action and to discuss its properties. In particular, we propose to construct at each point $\gamma \in \mathcal{M}$ a slice which is transversal to the orbit of \mathcal{D} through γ (definition of slice below). Each γ in \mathcal{M} defines a subgroup I_γ of \mathcal{D} called the isotropy group at γ --- $I_\gamma = \{\eta \in \mathcal{D} \mid A(\eta, \gamma) = \gamma\}$. We shall also show that if I_γ is trivial for some fixed γ , $I_{\gamma'}$ is also trivial for all γ' sufficiently near to γ in \mathcal{M} .

We first introduce some notation:*

Smooth means differentiable arbitrarily many times. M is a smooth manifold, $T(M)$ its tangent bundle, $T^*(M)$ the

* We use primarily the notation of 12.

cotangent bundle, $(T^*)^k$ the bundle of covariant k -tensors, and $S^k T^*$ the subbundle of symmetric covariant tensors. If $p \in M$ $T_p(M)$ is the tangent space to M at p ; similar notation for cotangent bundle, etc. If N is also a manifold and $f: M \rightarrow N$ is smooth, $Tf: T(M) \rightarrow T(N)$ is the induced map on the tangent bundle and $T_p f: T_p(M) \rightarrow T_{f(p)}(N)$ is defined by restricting Tf . $\langle \cdot, \cdot \rangle$ is a Riemannian metric, and $\langle \cdot, \cdot \rangle_p$ is the inner product on $T_p(M)$. If γ is a section of some vector bundle, γ_p is the value of γ in the fibre over p .

The right action $A: \mathcal{O} \times \mathcal{M} \rightarrow \mathcal{M}$ is defined as follows: If $\gamma \in \mathcal{M}$, $\eta \in \mathcal{O}$, then $A(\eta, \gamma)$ satisfies the equation $A(\eta, \gamma)_p(X, Y) = \gamma_{\eta(p)}(T_\eta X, T_\eta Y)$ for any $p \in M$, $X, Y \in T_p(M)$.

Our construction of a slice is essentially a generalization of the construction used for the action of a compact Lie group on a differentiable manifold (see 4, p. 108). Let M be a differentiable manifold, G a compact Lie group and $A: G \times M \rightarrow M$ a smooth map with the right action property stated above. Fix $x \in M$; let $G(x) = A(G, x)$, the orbit of G through x ; let G_x equal $\{g \in G | A(g, x) = x\}$, the isotropy group of x . Since A is continuous, G_x is closed in G , so G_x is a Lie subgroup of G . If $\pi: G \rightarrow G/G_x$ is the

projection map, there is a unique topology (the quotient topology) on G/G_x , such that π is continuous and open. With this topology G/G_x has a natural manifold structure such that $f: G/G_x \rightarrow \mathcal{R}$ is smooth if and only if $f \circ \pi: G \rightarrow \mathcal{R}$ is smooth. Also there exists a local cross section; that is, if U is a neighborhood of the trivial coset 0 in G/G_x there is a smooth map $\chi: U \rightarrow G$ such that $\chi(0) = \text{Id}$, the identity element of G , and $(\pi \upharpoonright \chi(U)) \circ \chi$ is the identity map on U (see 5, p. 110).

A induces a smooth injection $\phi: G/G_x \rightarrow M$ defined by $\phi(gG_x) = A(g, x)$. $T\phi$ is injective on every tangent space of G/G_x , so $\phi(G/G_x)$ is a submanifold of M , and $\phi: G/G_x \rightarrow \phi(G/G_x) = G(x)$ is a diffeomorphism, whose range $G(x)$ is compact and therefore closed in M .

Slice Theorem: (See 4, pp. 105, 108) Using the above notation, for any $x \in M$, there exists a submanifold S of M such that:

- 1) S is invariant under action of G_x
- 2) $A(g, S) \cap S \neq \emptyset$ implies $g \in G_x$
- 3) If $\chi: U \rightarrow G$ is a local cross section for G/G_x , and $F: U \times S \rightarrow M$ is defined by $F(u, s) = A(\chi(u), s)$, then F is a diffeomorphism onto an open set in M .

Outline of Proof: For $g \in G$, we will consider g as a map $g: M \rightarrow M$ defined by $g(x) = A(g, x)$. g is a smooth map with inverse the map defined by g^{-1} . Let $Tg: T(M) \rightarrow T(M)$ be the induced map on the tangent bundle.

Let M have a Riemannian metric, $\langle \cdot, \cdot \rangle$. By integrating it over G we can find a metric that is invariant under the action of G ; i.e., $g: M \rightarrow M$ becomes an isometry. Let Exp be the exponential map of M corresponding to the Riemannian structure. Then if $V \in T(M)$ such that $\text{Exp}(V)$ is defined, $\text{Exp}(Tg(V))$ is also defined, and $g \text{Exp}(V) = \text{Exp}(Tg(V))$.

$G(x)$ is a closed submanifold of M so the Riemannian structure on M defines a normal bundle $\nu(G(x))$ over $G(x)$, which is a subset of $T(M)$.

Since $G(x)$ is compact, one can find an $\varepsilon > 0$ such that if $N = \{V \in \nu(G(x)) \mid \langle V, V \rangle < \varepsilon^2\}$, $\text{Exp}(V)$ is defined for all V in N and $\text{Exp} \upharpoonright N$ is a diffeomorphism onto an ε -neighborhood of $G(x)$ in M (see 12, pp. 73-75). Let $\tilde{S} = N \cap T_x(M)$; that is \tilde{S} is the set of vectors at x with length less than ε , which are perpendicular to the subspace $T_x(G(x))$ of $T_x(M)$, and let $S = \text{Exp} \tilde{S}$. It is easy to check that S has the three properties of a slice (see 4, p. 108). Also, if $G_x = \{\text{Id}\}$, it follows that: $F: G \times S \rightarrow M$ by $F(g, s) = A(g, s)$ is a diffeomorphism

onto a neighborhood of $G(x)$ in M , and for any y in this neighborhood, $G_y = \{Id\}$.

When we try to generalize this theorem to the action of the group \mathcal{D} of diffeomorphisms of a manifold M , on the space of metrics \mathcal{M} of M , we encounter several difficulties. Our foremost problem is that \mathcal{D} is not compact, (nor is it locally compact). Therefore, we cannot integrate with respect to Haar measure to construct a metric on \mathcal{M} which is preserved under the action of \mathcal{D} . Also we do not know that an orbit $\mathcal{D}(x)$ of \mathcal{D} is compact, so we cannot immediately conclude that $\mathcal{D}(x)$ is closed in \mathcal{M} , or that the map $\phi: \mathcal{D} / \mathcal{D}_x \rightarrow \mathcal{D}(x)$ is a homeomorphism.

Furthermore there are technical difficulties regarding the topologies on the spaces \mathcal{D} and \mathcal{M} . If $C^\infty(S^2T^*)$ is the space of smooth symmetric covariant 2-tensors on M , and $C^\infty(S^2T^*)$ has the topology of uniform convergence in all derivatives, then it is a Frechet space. If we give \mathcal{M} this same topology, it is an open convex cone in $C^\infty(S^2T^*)$ --- an open set of a Frechet space. If we give \mathcal{D} the topology of uniform convergence in all derivatives, we find that \mathcal{D} is locally like a neighborhood in $C^\infty(T)$, the space of smooth vector fields, and this is also a Frechet space (see 5 or 6).

The usual proof of the slice theorem uses the fact that the manifold M has an exponential map, Exp , which is a diffeomorphism near the zero section of the tangent bundle. However, for Frechet spaces, the implicit function theorem and the local existence theorem for ordinary differential equations are not true in general (see 12, p. vi). Therefore we do not know that there is an exponential map with the above property, so we cannot use the usual proof with these spaces.

Because of these technical difficulties, we enlarge the sets \mathcal{M} and \mathcal{D} somewhat, so that they are infinite dimensional manifolds, locally like neighborhoods in Hilbert spaces.

These enlarged spaces, which we call \mathcal{D}^s and \mathcal{M}^s are spaces of H^s maps --- maps which have partial derivatives defined almost everywhere up to order s such that each partial is square integrable. A tangent space to \mathcal{D}^s or \mathcal{M}^s at any point looks like a Sobolev space, $H^s(E)$; i.e., a space of sections of some vector bundle E on M , such that each section has square integrable partial derivatives up to order s (see 16).

Also we can construct in a natural way, a Riemannian metric on \mathcal{M}^s which is invariant under the action of \mathcal{D} . The action A extends to a continuous map $A: \mathcal{D}^{s+1} \times \mathcal{M}^s \rightarrow \mathcal{M}^s$.

At any point $\gamma \in \mathcal{M}^S$, $\psi_\gamma: \mathcal{D}^{S+1} \rightarrow \mathcal{M}^S$ is defined by $\psi_\gamma(\eta) = A(\eta, \gamma)$. If γ is in \mathcal{M} , ψ_γ is smooth. Also, if for $\eta \in \mathcal{D}^{S+1}$, we identify the tangent spaces $T_\eta(\mathcal{D}^{S+1})$ and $T_{\psi_\gamma(\eta)}(\mathcal{M}^S)$ with the appropriate Sobolev spaces, $T_\eta \psi_\gamma$ becomes a first order linear differential operator α . It turns out that α has injective symbols so its range is closed in $T_{\psi_\gamma(\eta)}(\mathcal{M}^S)$.

The isotropy group \mathcal{D}_γ^{S+1} (also called I_γ) is a compact Lie group, and the factor space $\mathcal{D}^{S+1}/I_\gamma$ has an induced manifold structure. ψ_γ induces a map $\phi_\gamma: \mathcal{D}^{S+1}/I_\gamma \rightarrow \mathcal{M}^S$. ϕ_γ is an injective immersion, but we do not know that it is a homeomorphism onto a closed orbit (as in the classical case).

It does, however, induce a normal bundle ν on $\mathcal{D}^{S+1}/I_\gamma$, and the exponential, \exp on \mathcal{M}^S coming from the Riemannian structure of \mathcal{M}^S is defined near the zero section of ν , though it is not necessarily a diffeomorphism.

Also, we know that any fixed $\eta \in \mathcal{D}^{S+1}$ defines a smooth map $\eta^*: \mathcal{M}^S \rightarrow \mathcal{M}^S$ by $\eta^*(\gamma) = A(\eta, \gamma)$, and η^* is an isometry. Therefore, if \exp is defined on a vector V in $T(\mathcal{M}^S)$ \exp is defined on $T\eta^*(V)$, and $\eta^*\exp(V) = \exp T\eta^*(V)$.

Combining the above information, we get the following restricted version of the slice theorem.

Theorem: If $A: \mathcal{D}^{s+1} \times \mathcal{M}^S \rightarrow \mathcal{M}^S$ is the action defined above, for each $\gamma \in \mathcal{M}$, there is a submanifold S of \mathcal{M}^S such that:

- 1) If $\eta \in I_\gamma$ $A(\eta, S) = S$
- 2) There is a neighborhood V of I_γ in \mathcal{D}^{s+1} such that if $\eta \in V$ and $A(\eta, S) \cap S \neq \emptyset$ then $\eta \in I_\gamma$.
- 3) If $\chi: U \rightarrow \mathcal{D}^{s+1}$ is a local cross section for $\pi: \mathcal{D}^{s+1} \rightarrow \mathcal{D}^{s+1}/I_\gamma$, and $F: U \times S \rightarrow \mathcal{M}^S$ is defined by $F(u, s) = A(\chi(u), s)$, then F is a homeomorphism onto a neighborhood of γ .

In Section II, we develop some tools of calculus and discuss spaces of H^S functions on open sets in Euclidean space.

In Section III, we construct the infinite dimensional manifold of H^S maps from a compact manifold to some other manifold. We define the group \mathcal{D}^S and discuss its properties.

Section IV is about the manifold \mathcal{M}^S . Using differential operators we define a Riemannian metric μ on \mathcal{M}^S .

In Section V we discuss the group action $A: \mathcal{D}^{s+1} \times \mathcal{M}^S \rightarrow \mathcal{M}^S$, check its smoothness, and define a manifold structure on $\mathcal{D}^{s+1}/I_\gamma$.

In Section VI we show that $\phi: \mathcal{D}^{s+1}/I_\gamma \rightarrow \mathcal{M}^S$ is an injective immersion.

In Section VII we show that our Riemannian metric is invariant under the action of \mathcal{D}^{s+1} , and look at the normal bundle on $\mathcal{D}^{s+1}/I_\gamma$ induced by the immersion ϕ .

In Section VIII we state and prove the slice theorem itself.

In Section IX we discuss the smooth situation --- we show that $\phi_\gamma: \mathcal{D}/I_\gamma \rightarrow \mathcal{M}$ is a homeomorphism onto a closed subset of \mathcal{M} , and we prove that the set of metrics in \mathcal{M} with trivial isometry group is open, and dense in \mathcal{M} .

In Section X we make a few suggestions as to what further research might be done, extending the present work.

II. H^s Functions.

The purpose of this section is to define a set of functions from \mathbb{R}^n to \mathbb{R}^m , called H^s functions, (s a positive integer), and to put a topology on this set. Later we will generalize the definition to a set of maps from one manifold to another and construct two examples of such sets called \mathcal{D}^s and \mathcal{M}^s .

Let U be a bounded open set in \mathbb{R}^n . We shall construct $H^s(U)$, the H^s functions from U to \mathbb{R}^m and give $H^s(U)$ the structure of a Hilbert space.

Let $C^\infty(U)$ be the set of smooth functions from U to \mathbb{R}^m which are bounded and have all derivatives bounded, and let $C_0^\infty(U)$ be the subset consisting of those functions whose support is contained in U . These are linear spaces under the operations of pointwise addition and scalar multiplication. We shall define an inner product on $C^\infty(U)$.

Definition 1: If $f \in C^\infty(U)$, α an n -tuple of integers $(\alpha_1, \alpha_2, \dots, \alpha_n)$, and $|\alpha| = \sum_{i=1}^n \alpha_i$,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}, \text{ where } \{x_i\} \text{ are coordinate}$$

functions on \mathbb{R}^n .

Remark: $D^\alpha f \in C^\infty(U)$.

Definition 2: Let $\langle \cdot, \cdot \rangle_s$ be a bilinear function on $C^\infty(U)$ defined by:

$$(f, g)_s = \sum_{|\alpha| \leq s} \int_U D^\alpha f D^\alpha g \, dx$$

where dx refers to the usual measure on \mathbb{R}^n . It is clear that $(\cdot)_s$ is positive definite, so

$\|f\|_s = (f, f)_s^{1/2}$ defines a norm on $C^\infty(U)$.

Definition 3: $H^s(U)$ is the completion of $C^\infty(U)$ with respect to the norm $\|\cdot\|_s$ and $H_0^s(U)$ is the closure of $C_0^\infty(U)$ in $H^s(U)$ (see 2, p. 192). Both are Hilbert spaces in the norm $\|\cdot\|_s$. We shall write $H_0^s(U, \mathbb{R}^m)$ when we wish to explicitly specify the range.

We show an elementary property of the space $H_0^s(U)$.

Proposition 4: Let $U \subset V$ be bounded open sets of \mathbb{R}^n .

$U \subset V$. Then there is a natural inclusion

$i: H_0^s(U) \rightarrow H_0^s(V)$ defined by

$$i(f)(v) = \begin{cases} f(v) & v \in U \\ 0 & v \notin U \end{cases}$$

which is a continuous linear map.

Proof: $i: C_0^\infty(U) \rightarrow C_0^\infty(V)$ defined as above is a linear map. It is clearly bounded (in fact norm preserving) in the H^s norms on U and V . Therefore it extends to

the map $i: H_0^s(U) \rightarrow H_0^s(V)$. q.e.d.

Now let θ be any open set in \mathbb{R}^n and let $f: \theta \rightarrow \mathbb{R}^m$.

Definition 5: Given $p \in \theta$, we say that f is H^s at p iff there are bounded neighborhoods U, V of p with $V \subseteq U$, and a C^∞ function $\rho: U \rightarrow \mathbb{R}$ such that $\text{support } (\rho) \subseteq U$, ρ identically 1 on V and $\rho f \in H_0^s(U)$.

We now proceed to demonstrate some elementary properties of differentiation which we will need in our construction of spaces of H^s functions on manifolds.

Let U be an open set in the Banach space E and let $f: U \rightarrow F$, F a Banach space.

Definition 6: A function from \mathbb{R} to \mathbb{R} is called $O(t)$ iff $\lim_{t \rightarrow 0} O(t)/t = 0$.

Definition 7: Assuming U contains 0 , we say f is horizontal at 0 iff for any neighborhood V of 0 in F , there is a neighborhood U' of 0 in U such that

$$f(tU') \subseteq O(t)V$$

Definition 8: $f: U \rightarrow F$ is differentiable* at $p \in U$ if and only if there exists a continuous linear map $A: E \rightarrow F$

* This notion is often called the Frechet derivative.

such that $f(p + x) = f(p) + A(x) + h(x)$ for x near 0 where h is a horizontal function from some neighborhood of 0 in E into F . In this case A is called the derivative of f and is denoted $D_p f$.

Let $L(E, F)$ be the space of continuous linear maps from E to F . It is a Banach space under the norm
$$\|A\| = \max_{\|x\|=1} \|Ax\| \quad (\text{see 13, pp. 4-5}).$$
 Assume

$f: U \rightarrow F$ is differentiable at every point of U .

Definition 9: $Df: U \rightarrow L(E, F)$ is defined by $Df(p) = D_p f$.

If Df is continuous we say that f is C^1 . If Df

is differentiable at $p \in U$ we define $D^2 f = D(Df)$.

Proceeding inductively we write $D^k f = D(D^{k-1} f)$ and we

say f is C^k if and only if Df is C^{k-1} . Of course it

follows from this definition that f is C^k if and only if

$D^k f$ exists and is continuous. Also we say f is C^∞ or

smooth if f is C^k for all k . If f is C^k

$$D^k f: U \rightarrow L(E, L(E \cdots L(E, F)) \cdots) \quad (k - \text{L's}).$$

Let $L^k(E, F)$ be the set of continuous multi-linear maps from $E \times E \times \cdots \times E$ into F ($k - E$'s).

Definition 10: We define a norm on $L^k(E, F)$ by

$$\|A\| = \max_{|x_i|=1} \{A(x_1, x_2, \cdots, x_k)\} \quad \text{where } x_i \in E \text{ for all } i.$$

Proposition 11: $L^k(E, F)$ is a Banach space under this norm and there is a canonical norm-preserving isomorphism $Q: L^k(E, F) \rightarrow L(E, L(\dots (L(E, F)) \dots))$.

Proof: If $A \in L^k(E, F)$ let

$$(\dots ((Q(A)(x_1))(x_2)) \dots)(x_k) = A(x_1, x_2, \dots, x_k).$$

It is easy to see that Q is a norm-preserving isomorphism (see 13, p. 5).

Remark: Because of the above proposition we can identify $L(E, L(E, \dots L(E, F)) \dots)$ with $L^k(E, F)$ and consider Df as a map from U to $L^k(E, F)$.

Another concept of differentiability is the following:

Definition 12: $f: U \rightarrow F$ has Gateaux derivative at u in direction v iff $\lim_{t \rightarrow 0} \{\frac{1}{t}(f(u+tv) - f(u))\}$ exists in F . We write this limit as $df(u, v)$.

Lemma 13: If $f: U \rightarrow V$, $g: V \rightarrow F$ have Gateaux derivatives at x in direction v and at $f(x)$ in direction $df(x, v)$ respectively. $d(g \circ f)(x, v)$ exists and equals $dg(f(x), df(x, v))$.

Proof: Usual chain rule argument.

Lemma 14: If f has Gateaux Derivative $df(u, v)$ for all $u \in U$ and all $v \in E$ and df defines a continuous map

$\phi: U \rightarrow L(E, F)$ by $\phi(u)(v) = df(u, v)$, then f is C^1 on U and $\phi = Df$.

Proof: See 19, p. 6.

Let $d^k f(x, v_1 \cdots v_k)$ be the iterated k th Gateaux derivative.

Corollary 15: If f is C^{k-1} and $d^k f(x, v_1 \cdots v_k)$ exists for all $x \in U$, $\{v_i\} \subseteq E$ such that the map $\phi_k: U \rightarrow L^k(E, F)$ defined by $\phi_k(n)(v_1 \cdots v_k) = d^k f(u, v_1 \cdots v_k)$ is continuous, then f is C^k and $D_u^k f = \phi_k(u)$.

Proof: Assume the corollary for $k = t-1$, then $d^t f(u, v_1 \cdots v_t) = d(D^{t-1} f(v_1 \cdots v_{t-1}))(u, v_t)$. This by the lemma is $D_u(D^{t-1} f(v_1 \cdots v_{t-1}))(v_t)$ which is a bounded t -linear map in $\{v_i\}$. Therefore $D_u^t f$ exists and equals $d^t f(u, v_1 \cdots v_t)$. The case $t = 1$ is shown in the above lemma, so this proves the corollary.

This lemma and corollary will be used to show smoothness of maps from one space of functions to another.

This completes our discussion of the derivatives of a single map. We go on to develop the rules for differentiating a composition of mappings.

Proposition 16: Let U, V be open sets of E, F ; $f: U \rightarrow V$, $g: V \rightarrow G$ (E, F, G Banach spaces). Then if

f and g are C^1 , so is $g \circ f$ and

$D_p(g \circ f) = D_{f(p)}g \cdot D_p f$ where " \cdot " indicates the composition of linear maps $D_p f: E \rightarrow F$ and $D_{f(p)}g: F \rightarrow G$.

Proof: Standard (see 17, p. 17).

Let $E \oplus F$ be the direct sum of E and F (a Banach space under the norm $\|e + f\| = (\|e\|^2 + \|f\|^2)^{1/2}$).

Definition 17: If $f: G \rightarrow E$, $g: G \rightarrow F$, then

$f \oplus g: G \rightarrow E \oplus F$ is defined by $(f \oplus g)(x) = f(x) + g(x)$.

Proposition 18: If f and g (above) are differentiable at a point p of E , $f \oplus g$ is differentiable at p , and $D_p(f \oplus g) = D_p f \oplus D_p g$.

Proof: First we note that the sum of two horizontal maps is horizontal. Let $h_1: G \rightarrow E$, $h_2: G \rightarrow F$ be defined and horizontal at 0 . Then given V a neighborhood of 0 in $E \oplus F$, there exists neighborhoods V_1 and V_2 of 0 in E and F respectively such that $V_1 \oplus V_2 \subseteq V$. Since h_1, h_2 are horizontal there exist neighborhoods U_1 and U_2 of 0 in G , such that $h_1(tU_1) \subseteq 0(t)V_1$, $h_2(tU_2) \subseteq 0(t)V_2$. Let $U = U_1 \cap U_2$. Then $h_1 \oplus h_2(tU) \subseteq 0(t)(V_1 \oplus V_2) \subseteq 0(t)V$.

$h_1 \oplus h_2$ is horizontal.

Our proposition now follows from the computation:

$$\begin{aligned}
 (f \odot g)(p + x) &= f(p + x) + g(p + x) \\
 &= f(p) + D_p f(x) + h_1(x) + g(p) + D_p g(x) + h_2(x) \\
 &= (f \odot g)(p) + (D_p f \odot D_p g)(x) + (h_1 \odot h_2)(x) .
 \end{aligned}$$

We now prove a slight generalization of the formula for the derivative of the product of two functions. Let E, F, G be Banach spaces; let $g: E \times F \rightarrow G$ be a continuous bilinear map. Let U, V be open sets in Banach spaces A, B respectively, and let $f: U \rightarrow E$ $h: V \rightarrow F$ be C^k functions.

Proposition 19: $g(f, h): U \times V \rightarrow G$ defined by $g(f, h)(u, v) = g(f(u), h(v))$ is C^k , and $D_{(u, v)}(g(f, h))(a, b) = g(D_u f(a), h(v)) + g(f(u), D_v h(b))$ where $a \in A$, $b \in B$, $u \in U$, $v \in V$.

Proof: This is analogous to the standard proof for the derivative of a product (see 5a, p. 167-169).

Let a map $g: L(E, F) \times L(F, G) \rightarrow L(E, G)$ be defined by composition, $g(X, Y) \rightarrow X \cdot Y$, and assume $f: U \rightarrow V$, $h: V \rightarrow G$ are C^2 where U and V are open sets of E and F respectively.

Corollary 20: $D_u^2(h \circ f) = D_{f(u)}^2(D_u f, D_u f) + D_{f(u)} h \cdot D_u^2 f$.

Proof: g is a continuous bilinear mapping, so use the

above proposition on $D_u(h \circ f) = D_{f(u)}h \cdot D_u f$ and the composition rule to get $D(D_{f(u)}h) = D_{f(u)}^2 h \cdot D_u f$.

Corollary 21: Let the above f and h be C^k . Then $D_u^k(h \circ f)$ is a sum of terms of the form:

$$D_{f(u)}^p h \cdot (D_u^{i_1} f, D_u^{i_2} f, \dots, D_u^{i_p} f)$$

where $i_1 + i_2 + \dots + i_p = k$.

Proof: We have the case $k = 2$. The proof for larger k is by a direct induction using the preceding Corollary and the two previous propositions.

The following Corollary will be useful in studying the composition of H^s functions.

Corollary 22: $D^s(h \circ f)$ is a sum of terms in $D^t f$ and $D^r h$ such that for each term, t or r is less than or equal to $[s/2]^* + 1$.

Proof: This follows by inspecting the formula of the preceding corollary.

If we apply the arguments similar to these of the above three corollaries to the situation $f: U \rightarrow E$, $h: V \rightarrow F$, $g: E \times F \rightarrow G$, g a bilinear map. We get:

*[a] means the greatest integer less than or equal to a .

Proposition 23: $D^s g(f, h) = \sum_{i=0}^s \binom{s}{i}^* g(D^i f, D^{s-i} h) .$

Proof: $D g(D^i f, D^j h) = g(D^{i+1} f, D^j h) + g(D^i f, D^{j+1} h)$

by the previous proposition. The formula now follows from the binomial theorem.

Corollary 23a: $D^s(g \circ (f, h))$ is a sum of terms of form $g(D^r f, D^t h)$ where r or $t \leq [s/2] + 1$.

If we now restrict our attention to the finite dimensional situation --- let U, V be open in \mathbb{R}^m and \mathbb{R}^n , $f: U \rightarrow V$, $h: V \rightarrow \mathbb{R}^p$ ---, then $D_u f, D_{f(u)} h$ are linear transformations represented by the matrix of first order partial derivative of f , and h at u and $f(u)$ respectively, (see 5a, pp. 170-171). More generally $D_u^r f$ is a matrix of r^{th} order partial derivatives of f . Corollary 22 implies the following result:

Proposition 24: $D_u^s(h \circ f)$ is a matrix of polynomials in the partial derivatives of f and h , of order less than or equal to s . Each term of such a polynomial will be $N D^\alpha h D^\beta f$ where N is an integer and $|\alpha| \leq [s/2] + 1$ or $|\beta| \leq [s/2] + 1$.

Similarly, Corollary 23a implies:

* $\binom{s}{i}$ is the binomial coefficient $\binom{s}{i} = \frac{s!}{i!(s-i)!}$

Proposition 25: $D_{(u,v)}^s(g(f,h))$ is a matrix of polynomials in the partial derivatives of f and h . Each term of such a polynomial will be $P(g) D^\alpha f D^\beta h$ where $P(g)$ is a function of the coefficients in the matrix which represents the bilinear transformation g , and where $|\alpha| \leq [s/2] + 1$ or $|\beta| \leq [s/2] + 1$.

We now state an important property of $H_0^s(U)$, which compares H^s functions and C^k functions, and which, when combined with the previous proposition, allows us to define H^s functions on manifolds.

U is an open set of \mathbb{R}^n . Let $C^k(U)$ be the set of C^k functions from U to \mathbb{R}^m which are bounded, and whose first k derivatives are bounded. It is a linear space under pointwise operations.

Proposition 26: $C^k(U)$ is complete under the norm

$$\|f\|_k = \max_{u \in U} \left\{ \sum_{i=0}^k |D_u^i f| \right\} \quad (\text{where } D^0 f = f).$$

Proof: Standard, (see 1, pp. 4-5).

We will endow $C^k(U)$ with the topology induced by the above norm. It is of course the topology of "uniform C^k convergence".

Lemma 27: (Sobolev)

If $s \geq [n/2] + k+1$, $H_0^s(U) \subseteq C^k(U)$ and the inclusion

is a continuous linear map.

Proof: See 2, p. 194 or 20, pp. 465-467.

Remark: The above lemma shows that if a function $f: U \rightarrow \mathbb{R}^n$ is H^s , then it is C^k , if $s \geq [n/2] + k + 1$. This is so because we know that, for all $p \in U$, there exists some function ρ , such that $\rho f \in H_0^s(U')$, $p \in U' \subseteq U$ and $\rho f = f$ in a neighborhood of p . Therefore, since ρf is C^k at p , so is f .

We now discuss the properties of composition of H^s functions.

Composition Lemma 28: Let U, V be bounded open sets of Euclidean space $g: V \rightarrow U$, $g \in H^s(V)$ and $f \in H_0^s(U)$. Assume s large enough so that $H_0^{[s/2]}(U) \subseteq C^1(U)$.

$H_0^{[s/2]}(V) \subseteq C^1(V)$, as in Sobolev lemma. Also assume $D_v g$ is non-singular at each point v of V . If $\{f_n\} \subseteq C_0^\infty(U)$, $\{g_n\} \subseteq C^\infty(V)$, $(g_n(V) \subseteq U)$, such that $f_n \rightarrow f$ in $H_0^s(U)$ and $g_n \rightarrow g$ in $H^s(V)$ and in $C^{[s/2]+1}(V)$ then $f_n g_n \rightarrow fg$ in $H^s(V)$.

Proof: We use proposition 24 combined with the Sobolev lemma. We know that since f and g are C^0 , fg is C^0 , and clearly $f_n g_n \rightarrow fg$ in $C^0(U)$ (see 1, pp. 7-8). Therefore, we must only show that $f_n g_n$ converges in $H^s(V)$.

That is, we must show that for all $k \leq s$, each entry of the matrix $D^k(f_n g_n)$ converges in the L_2 sense on V .

By proposition 24, any such entry is: $N D^{\alpha} f_n D^{\beta} g_n$, where $|\alpha| \leq [s/2] + 1$ or $|\beta| \leq [s/2] + 1$.

Case I:

If $|\alpha| \leq [s/2] + 1$, $D^{\alpha} f_n \rightarrow D^{\alpha} f$ in $H^{s-[s/2]+1}(U) \subseteq H^{[s/2]-1}(U) \subseteq C^0(U)$ by hypothesis.

Also, $D^{\beta} g_n \rightarrow D^{\beta} g$ in L_2 sense. Therefore

$$\begin{aligned} \int_V |D^{\alpha} f_n D^{\beta} g_n - D^{\alpha} f D^{\beta} g|^2 &\leq \int_V |D^{\alpha} f_n D^{\beta} g_n - D^{\alpha} f D^{\beta} g_n|^2 \\ &\quad + \int_V |D^{\alpha} f D^{\beta} g_n - D^{\alpha} f D^{\beta} g|^2 \\ &\leq \max_{u \in U} \{|D_u^{\alpha} f_n - D_u^{\alpha} f|^2\} \int_V |D^{\beta} g_n|^2 \\ &\quad + \max_{u \in U} \{|D_u^{\alpha} f|^2\} \int_V |D^{\beta} g_n - D^{\beta} g|^2 \end{aligned}$$

$\int_V |D^{\beta} g_n|^2$ is bounded because $D^{\beta} g_n \rightarrow D^{\beta} g$ in L_2 , and

$\int_V |D^{\beta} g_n - D^{\beta} g|^2$ goes to zero for the same reason.

$\max_{u \in U} \{|D_u^{\alpha} f|^2\}$ is bounded because $D^{\alpha} f_n \rightarrow D^{\alpha} f \in C^0$.

$\max_{u \in U} \{|D_u^{\alpha} f_n - D_u^{\alpha} f|^2\}$ goes to zero for the same reason.

Hence, if $|\alpha| \leq [s/2] + 1$, $D^{\alpha} f_n D^{\beta} g_n$ converges.

Case II:

$|\beta| \leq [s/2] + 1$. As in Case I, we need to show:

$$\int_V |D_{g_n(v)}^{\alpha} f_n - D_{g_n(v)}^{\beta} f_n|^2 + \int_V |D_{g_n(v)}^{\alpha} f_n - D_{g(v)}^{\beta} f_n|^2$$

$$\rightarrow 0$$

$$\begin{aligned} \int_V |D_{g_n(v)}^{\alpha} f_n - D_{g_n(v)}^{\beta} f_n|^2 &\leq \max_{v \in V} \{|D_{g_n(v)}^{\beta} f_n|^2\} \int_V |D_{g_n(v)}^{\alpha} f_n \\ &\quad - D_{g(v)}^{\alpha} f_n|^2 \end{aligned}$$

$$\begin{aligned} \int_V |D_{g_n(v)}^{\alpha} f_n - D_{g(v)}^{\beta} f_n|^2 &\leq \max_{v \in V} \{|D_{g_n(v)}^{\beta} f_n - D_{g(v)}^{\beta} f_n|^2\} \int_V |D_{g_n(v)}^{\alpha} f_n|^2 \\ &\quad + \max_{v \in V} \{|D_{g_n(v)}^{\beta} f_n|^2\} \int_V |D_{g_n(v)}^{\alpha} f_n - D_{g(v)}^{\alpha} f_n|^2 \end{aligned}$$

The first "max" term is bounded, the second goes to zero and the third is bounded as in case I.

However, for the integral terms we must use the fact that Dg_n is non-singular. $g_n \in C^1(V)$ so Dg_n is bounded. Hence we can assume that $\frac{1}{\det(Dg_n)}$ (the reciprocal of the determinant) is also bounded on V , and the bound is uniform in n .

We know $\int_U |D_u^{\alpha} f|^2 < \infty$ and

$$\int_U |D_u^{\alpha} f|^2 = \int_{g^{-1}(U)} |D_{g(v)}^{\alpha} f|^2 |\det D_v g| \geq \int_V |D_{g(v)}^{\alpha} f|^2 |\det D_v g| .$$

$$\text{Hence } \int_V |D_{g(v)}^{\alpha} f|^2 \leq \max_{v \in V} \{|\det D_v g|^{-1}\} \int_U |D_u^{\alpha} f|^2$$

$$\int_V |D_{g(v)}^{\alpha} f|^2 < \infty .$$

Similarly $\int_V |D_{g_n(v)}^{\alpha} f_n - D_g^{\alpha}(v) f|^2$

$$\leq \max_{v \in V} \{ |\det D_v g|^{-1} \} \int_U |D_u^{\alpha} f_n - D_u^{\alpha} f|^2$$

$$+ (\max_{v \in V} \{ |\det D_v g_n|^{-1} \} - \max_{v \in V} \{ |\det D_v g|^{-1} \}) \int_U |D_u^{\alpha} f_n|^2$$

and $\int_V |D_{g_n(v)} f - D_g(v) f| \leq (\max_{v \in V} \{ |\det D_v g_n|^{-1} \})$

$$- \max_{v \in V} \{ |\det D_v g|^{-1} \}) \int_U |D_u^{\alpha} f|^2 .$$

so these terms go to zero. The lemma follows.

From here on, when dealing with H^s functions, we shall always assume that s is large enough so that the above lemma holds.

Remarks: 1) $f_n g_n \rightarrow fg$ in $H^s(V)$ means that $fg \in H^s(V)$ because $H^s(V)$ is complete. Therefore, since the definition of " H^s function" is a purely local one (we will prove this soon) we have shown that $f, g \in H^s$ at $g(p)$ and p implies $f \circ g \in H^s$ at p whenever $D_p g$ is non-singular.

2) The proof of the above lemma shows that for $k \leq s$, $D^k(f_n g_n)$ converges to $D^k(fg)$ in the L_2 sense. Since $D^k f_n \xrightarrow{L_2} D^k f$ and the same for g , we find that the composition rule for derivatives holds almost everywhere for the function $f \circ g$.

3) If $g: V \rightarrow U$ has the property that any neighborhood of the boundary of U contains the image under g of a neighborhood of the boundary of V , then for all $f \in H_0^s(U)$, $g \circ f \in H_0^s(V)$.

4) If $f_n \rightarrow f$ in $C^s(U)$, we do not need the hypothesis that Dg is non-singular because then, for all $k \leq s$, $D^k f$ is bounded and $D^k f - D^k f_n$ goes uniformly to zero on U . Hence $\int_V |D_{g(V)}^k f|^2 < \infty$ and

$$\int_V |D_{g(v)}^k f - D_{g(v)}^k f_n|^2 \rightarrow 0 .$$

5) If $f \in H_0^S(U)$ we can say that the derivatives of f are defined almost everywhere on U , they being the L_2 limit of the corresponding derivatives of f_n where $\{f_n\} \subseteq C_0^\infty(U)$ and $f_n \rightarrow f$ in $H_0^S(U)$.

We now prove another property of H^S functions which is also a consequence of the Sobolev lemma and proposition 25.

Lemma 29: Let $g: E \times F \rightarrow G$ be a bilinear map; E, F , and G are Euclidean spaces. Let $e \in H_0^S(U, E)$, $f \in H_0^S(U, F)$. Let $\Delta: U \rightarrow U \times U$ be the diagonal map. Then $g \circ (e, f) \circ \Delta \in H_0^S(U, G)$ and the map $(e, f) \rightarrow g \circ (e, f) \circ \Delta$ is a continuous bilinear map $B: H_0^S(U, E) \times H_0^S(U, F) \rightarrow H_0^S(U, G)$.

Proof: Let $e_n \rightarrow e$ in $H_0^S(U)$, $f_n \rightarrow f$ in $H_0^S(V)$, where $\{e_n\} \subseteq C_0^\infty(U)$, $\{f_n\} \subseteq C_0^\infty(V)$.

Claim: $\{g(e_n, f_n) \circ \Delta\} \subseteq C_0^\infty(U)$. Since g is smooth and Δ is smooth, it is clear that $g(e_n, f_n) \circ \Delta$ is smooth, so we need only show that $\text{support}(g \circ (e_n, f_n) \circ \Delta) \subseteq U$. But if $K_1 = \text{support}(e_n)$, $K_2 = \text{support}(f_n)$, then $\text{support}(g \circ (e_n, f_n) \circ \Delta) \subseteq K_1 \cap K_2$. Therefore $(g \circ (e_n, f_n) \circ \Delta) \subseteq C_0^\infty(U)$.

Now we wish to show that

$$g \circ (e_n, f_n) \circ \Delta \rightarrow g \circ (e, f) \circ \Delta \text{ in } H_0^s(U) .$$

$$D\Delta: U \rightarrow L(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n) \text{ by } D_u \Delta = \Delta: \mathbb{R}^n \times \mathbb{R}^n \text{ and}$$

$D_u^2 \Delta = 0$, the bilinear map with image $\{0\}$. Therefore, by the composition rule for derivatives and proposition 19,

$$D_u(g \circ (e, f) \circ \Delta) = g(D_u e, f) + g(e, D_u f) \text{ and more generally}$$

$$D^s(g \circ (e, f) \circ \Delta) = D^s(g \circ (e, f)) \circ \Delta . \text{ By proposition 25,}$$

$D_u^s(g \circ (e, f))$ is a matrix of polynomials whose terms

have form $p(g) D_u^\alpha e D_u^\beta f$ such that $|\alpha| \leq [s/2] + 1$ or

$|\beta| \leq [s/2] + 1$. Therefore, to show

$$g(e_n, f_n) \circ \Delta \rightarrow g(e, f) \circ \Delta \text{ in } H_0^s(U) \text{ we need only show}$$

$$x_{\alpha\beta}^n = \int_U |p(g) D^\alpha e_n D^\beta f_n - p(g) D^\alpha e D^\beta f|^2 \rightarrow 0 . \text{ But}$$

$$x_{\alpha\beta}^n \leq |p(g)|^2 \int_U |D^\alpha e_n D^\beta f_n - D^\alpha e D^\beta f_n|^2$$

$$+ |p(g)|^2 \int_U |D^\alpha e D^\beta f_n - D^\alpha e D^\beta f|^2 .$$

Therefore, if $|\alpha| \leq [s/2] + 1$,

$$x_{\alpha\beta}^n \leq |p(g)|^2 \max_{u \in U} \{|D_u^\alpha e_n - D_u^\alpha e|^2\} \int_U |D^\beta f_n|^2$$

$$+ |p(g)|^2 \max_{u \in U} \{|D_u^\alpha e|^2\} \int_U |D^\beta f_n - D^\beta f|^2 .$$

From here it follows as in the composition lemma that

$$x_{\alpha\beta}^n \rightarrow 0 .$$

$x_{\alpha\beta}^n$ is symmetric in the triplets (α, e, e_n) and (β, f, f_n) . Therefore, $x_{\alpha\beta}^n \rightarrow 0$ in the case

$|\beta| \leq [s/2] + 1$ also. Hence $g(e, f) \circ \Delta \in H_0^s(U)$.

Now we check that $B: H_0^s(U) \times H_0^s(U) \rightarrow H_0^s(U)$ is a bilinear continuous map. B is obviously bilinear, because g is.

To check continuity we must show that there is a constant K such that $\|B(e, f)\|_s \leq K \|e\|_s \|f\|_s$. By proposition 25 and the definition of H^s norms it is enough to show:

There is a constant $K_{\alpha\beta}$ such that $\int_U |p(g)| |D^\alpha e D^\beta f|^2 \leq K_{\alpha\beta} \|e\|_s \|f\|_s$ where $|\alpha| \leq [s/2] + 1$ or $|\beta| \leq [s/2] + 1$. If $|\alpha| \leq [s/2] + 1$,

$$\int_U |p(g)| |D^\alpha e D^\beta f|^2 \leq |p(g)|^2 \max_{u \in U} |D_u^\alpha e|^2 \int_U |D^\beta f|^2$$

$\int_U |D^\beta f|^2 \leq \|f\|_s$ and by Sobolev lemma, there is a constant C , such that $\max_{u \in U} |D_u^\alpha e| \leq C \|e\|_s$.

Therefore let $K_{\alpha\beta} = |p(g)| C$. By symmetry we can find $K_{\alpha\beta}$ in the case $|\beta| \leq [s/2] + 1$ as well. The lemma follows.

Corollary 30: If $e \in H_0^s(U, \mathbb{R}^{m \times n})$ and $f \in H_0^s(U, \mathbb{R}^{n \times p})$

let $B(e, f)$ be defined by $B(e, f)(u) = e(u) \cdot f(u)$

("." means matrix multiplication). Then

$B: H_0^s(U, \mathbb{R}^{m \times n}) \times H_0^s(U, \mathbb{R}^{n \times p}) \rightarrow H_0^s(U, \mathbb{R}^{n \times p})$ is a

continuous bilinear map.

Remark: This corollary shows us that the definition of a map's being H^s at a point p , is local. That is, assume U, V are neighborhoods of p such that $p \in C_0^\infty(U)$, $\rho \equiv 1$ on V and $\rho f \in H_0^s(U)$. Now let U' be any neighborhood of p which is included in U . There exists a neighborhood $V' \subseteq U'$ and $\rho' \in C_0^\infty(\rho')$ such that $\rho' \upharpoonright V' \equiv 1$. Then $\rho' \in C_0^\infty(U)$ (extending by 0 outside U') so $\rho' \in H_0^s(U)$ and $\rho\rho'f \in H_0^s(U)$ by the corollary, also $\rho\rho' \equiv 1$ on $V \cap V'$. So $V \cap V'$ and U' can be used to show that f is H^s .

Corollary 31: Using the notation of the above lemma, if $e \in H^s(U, E) \cap C^{[s/2]+1}(U)$, then there is a continuous linear map $\mu_e: H_0^s(U, F) \rightarrow H_0^s(U, G)$ defined by $\mu_e(f) = g \circ (e, f) \circ \Delta$.

Proof: We need only note that the proof of lemma 29 required L_2 bounds on all s derivatives of functions in $H^s(U, E)$ and uniform bounds on the first $[s/2] + 1$ derivatives. Therefore the proof works for this corollary as well.

This concludes our discussion of the elementary properties of H^s functions. We now go on to investigate further the effect of composition of maps on the H^s spaces.

α -Lemma 32: Let U, V, g be as in the composition lemma.

Let $\alpha_g: H_0^s(U) \rightarrow H^s(V)$ be defined by $\alpha_g(f) = f \circ g$.

α_g is a continuous linear mapping, and it is therefore smooth.

Proof: That α_g is linear, is obvious, so we need only show that α_g is bounded. For this it is sufficient to find for all γ such that $|\gamma| \leq s$ a constant K_γ such that $\int_V |D^\gamma(fg)|^2 \leq K_\gamma \sum_{|\eta| \leq s} \int_U |D^\eta f|^2 = K_\gamma \|f\|_s^2$. By

proposition 24, we know that to do this we need only find constants $K_{\alpha\beta}$ such that $\int_V |D^\alpha f \cdot D^\beta g|^2 \leq K_{\alpha\beta} \|f\|_s^2$, and we can assume

$$|\alpha| \leq [s/2] + 1 \quad \text{or} \quad |\beta| \leq [s/2] + 1.$$

Case I: $|\alpha| \leq [s/2] + 1$.

$$\begin{aligned} \int_V |D^\alpha f \cdot D^\beta g|^2 &\leq \max_{u \in U} \{ |D^\alpha f|^2 \} \int_V |D^\beta g|^2 \\ &\leq Q^2 \int_V |D^\beta g|^2 \|f\|_s^2 \end{aligned}$$

where Q is the bound on the continuous linear injection $H_0^s(U) \subseteq C^{|\alpha|}(U)$. Therefore, let $K_{\alpha\beta} = Q^2 \int_V |D^\beta g|^2$.

Case II: $|\beta| \leq [s/2] + 1$.

$$\begin{aligned} \int_V |D^{\alpha} f D^{\beta} g|^2 &\leq \max_{v \in V} |D_v^{\beta} g|^2 \int_V |D_{g(v)}^{\alpha} f|^2 \\ &\leq \max_{v \in V} \{|D_v^{\beta} g|^2\} \max_{v \in V} \{|\det D_v g|^{-1}\} \int_U |D_u^{\alpha} f|^2 \\ &\leq \max_{v \in V} \{|D_v^{\beta} g|^2\} \max_{v \in V} \{|\det D_v g|^{-1}\} \|f\|_s^2. \end{aligned}$$

Therefore, let $K_{\alpha\beta} = \max_{v \in V} \{|D_v^{\beta} g|^2\} \max_{v \in V} \{|\det D_v g|^{-1}\}$.

The lemma follows.

As a dual to the above lemma we have the following:

ω -Lemma 33: Let V be an open set in \mathbb{R}^n and $g \in C^{\infty}(V)$ with values in \mathbb{R}^m ; i.e., for all k , $g \in C^k(V)$. Let $H_0^s(U, V) = \{f \in H_0^s(U) \mid f(U) \subseteq V\}$. Then $H_0^s(U, V)$ is open in $H_0^s(U, \mathbb{R}^n)$ and $\omega_g: H_0^s(U, V) \rightarrow H^s(U, \mathbb{R}^m)$ defined by $f \rightarrow g \circ f$ is a smooth map.

Proof: If V is open in \mathbb{R}^n , it is clear that $C^0(U, V) = \{f \in C^0(V) \mid f(U) \subseteq V\}$ is open in $C^0(U)$, because $C^0(U)$ has the uniform convergence topology. But $H_0^s(U) \subseteq C^0(U)$ is a continuous linear injection and $H_0^s(U, V) = H_0^s(U) \cap C^0(U, V)$. Therefore $H_0^s(U, V)$ is open in $H_0^s(U, \mathbb{R}^n)$. $g \circ f \in H^s(U, \mathbb{R}^m)$ by Remark 3 following the composition lemma, so the image of ω_g is in $H^s(U, \mathbb{R}^m)$.

We shall show the smoothness of ω_g in a number of steps, as follows.

- 1) ω_g continuous.
- 2) The Gateaux derivative $d\omega_g(f,h)$ exists for all $f \in H_0^s(U,V)$, $h \in H_0^s(U, \mathbb{R}^n)$.
- 3) $d\omega_g(f,h)(p) = D_f(p)(h(p))$.
- 4) Therefore, Gateaux derivative is continuous as map from $H_0^s(U,V)$ to $L(H_0^s(U, \mathbb{R}^n), H^s(U, \mathbb{R}^m))$.
- 5) Therefore, $D\omega_g$ exists and $D_f\omega_g(h) = d\omega_g(f,h)$.
- 6) ω_g is C^1 and by induction, smooth.

Proof of steps:

- 1) Let $f_n \rightarrow f$ in $H_0^s(U,V)$; we must show $g f_n \rightarrow g f$ in $H^s(U, \mathbb{R}^m)$. As in the composition and α lemmas, this means that we must show

$$\int_U |D^\alpha g D^\beta f_n - D^\alpha g D^\beta f|^2 \rightarrow 0 .$$

$$\text{But } \int_U |D_{f_n}^\alpha g D^\beta f_n - D^\alpha g D^\beta f|^2 \leq \max_{v \in V} |D_v^\alpha g|^2 \int_U |D^\beta f_n - D^\beta f|^2$$

$$\max_{v \in V} |D_v^\alpha g|^2 < \infty \text{ since each derivative of } g \text{ is bounded,}$$

$$\text{and } \int_U |D^\beta f_n - D^\beta f|^2 \rightarrow 0 \text{ since } f_n \rightarrow f \text{ in } H_0^s(U,V) .$$

- 2) By definition $d\omega_g(f,h) = \lim_{t \rightarrow 0} \frac{1}{t}(g(f+th) - g(f))$, the limit being in the topology of $H^s(U, \mathbb{R}^m)$.

Therefore it is sufficient to show that

$$X_t = \frac{1}{t}(D_{f+th}^{\alpha} g D^{\beta}(f+th) - D_f^{\alpha} g D^{\beta} f) \text{ converges in the } L_2$$

sense for $|\alpha|$ and $|\beta| < s$.

But $X_t = \frac{1}{t}(D_{f+th}^{\alpha} g - D_f^{\alpha} g)D^{\beta} f + D_{f+th}^{\alpha} g D^{\beta} h$. Since s is large, h is bounded in the C^0 norm. By the mean value theorem, for all $u \in U$, $\frac{1}{t}(D_{f(u)+th(u)}^{\alpha} g - D_{f(u)}^{\alpha} g) = D(D_{f(u)}^{\alpha} g)(h(u))$ for some $v \in V$, such that $v = f(u) + t'h(u)$, $0 \leq t' \leq t$. But since $g \in C^{\infty}(V, \mathbb{R}^m)$, $D(D_{f(u)}^{\alpha} g)$ is uniformly continuous as a function of v .

Therefore, since h is uniformly bounded, we know $\frac{1}{t}(D_{f+th}^{\alpha} g - D_f^{\alpha} g)$ converges to $D(D_f^{\alpha} g)(h)$ uniformly on U . Hence $\frac{1}{t}(D_{f+th}^{\alpha} g - D_f^{\alpha} g)D^{\beta} f \rightarrow (D(D_f^{\alpha} g)(h))D^{\beta} f$ in the L_2 sense because $D^{\beta} f$ is L_2 on U .

Now for the second term of X_t . $g \in C^{\infty}(V, \mathbb{R}^m)$ and h bounded in C^0 norm implies $D_{f+th}^{\alpha} g \rightarrow D_f^{\alpha} g$ uniformly on U as above. Therefore $D_{f+th}^{\alpha} g D^{\beta} h \rightarrow D_f^{\alpha} g D^{\beta} h$ in L_2 sense, so $d\omega_g(f, h)$ exists.

$$3) \frac{1}{t}(g(f(u) + th(u)) - g(f(u))) = D_{f(u)+t'h(u)} g(h(u)),$$

by the mean value theorem, where $0 < t' < t$. But $D_v g$ is uniformly continuous in V and h is bounded on U , so $\frac{1}{t}(g(f(u) + th(u)) - g(f(u))) \rightarrow D_{f(u)} g(h(u))$ uniformly on U . By 2), $\frac{1}{t}(g(f + th) - g(f))$ converges in

$H_0^S(U, \mathbb{R}^m)$. By the above, it converges to $D_f g(h)$ in $C^0(U, \mathbb{R}^m)$ or $H^0(U, \mathbb{R}^m)$. Therefore it converges to $D_f g(h)$ in $H_0^S(U, \mathbb{R}^m)$ or $d\omega_g(f, h) = D_f g(h)$.

4) $Dg: V \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is smooth and has bounded derivatives. Therefore the map

$\omega_{Dg}: H_0^S(U, V) \rightarrow H^S(U, L(\mathbb{R}^n, \mathbb{R}^m))$ defined by $f \rightarrow D_f g$ is continuous by step 1) . Let

$\theta: H_0^S(U, L(\mathbb{R}^n, \mathbb{R}^m)) \rightarrow L(H_0^S(U, \mathbb{R}^n), H^S(U, \mathbb{R}^m))$ defined by $\theta(g)(f)(u) = (g(u))(f(u))$. It is immediate that θ is a continuous linear map (use Corollary 30). Also, the map induced by $d\omega_g$ from $H_0^S(U, V)$ to $L(H_0^S(U, \mathbb{R}^n), H^S(U, \mathbb{R}^m))$ is $\theta \circ \omega_{Dg}$.

5) Therefore, $d\omega_g$ defines the continuous map $\theta \circ \omega_{Dg}$, so by lemma 13, $D\omega_g$ exists and is continuous, and $D_f \omega_g(h) = d\omega_g(f, h)$.

6) By the 5 steps above, $\omega_{Dg}: H_0^S(U, V) \rightarrow H^S(U, L(\mathbb{R}^n, \mathbb{R}^m))$ is C^1 . Also θ defined above, induces a continuous linear map

$$\tilde{\theta}: L(H_0^S(U, \mathbb{R}^n), H^S(U, L(\mathbb{R}^n, \mathbb{R}^m))) \rightarrow L(H_0^S(U, \mathbb{R}^n), L(H_0^S(U, \mathbb{R}^n), H^S(U, \mathbb{R}^m)))$$

defined by $\tilde{\theta}(\ell) = \theta \circ \ell$. $D^2\omega_g = \tilde{\theta} \circ D\omega_{Dg}$. Hence ω_g

is C^2 . The existence of the k^{th} derivative follows in the same way using $D\omega_{D^{k-1}g}$ and

$$D_f^k \omega_g(h_1, \dots, h_k) = D_f^k g(h_1, \dots, h_k) \quad \text{q.e.d.}$$

Weak ω -Lemma 34: Use notation of the ω -lemma except let g be in $H^s(V) \cap C^{[s/2]+1}(V)$. Let A be a subset of $H^s(U, V)$ such that for all $f \in A$, $D_p f$ is non-singular for every p in U and $f \in C^{[s/2]+1}(U)$. Then

$$\omega_g: A \rightarrow H^s(U, \mathbb{R}^n) \quad \text{and if } f_n \rightarrow f \text{ in } H^s(U, V) \text{ and in } C^{[s/2]+1}(U), \quad \omega_g(f_n) \rightarrow \omega_g(f).$$

Proof: $\omega_g(A) \subseteq H^s(U, \mathbb{R}^n)$ by the composition lemma.

The convergence property follows from the estimates of step 1) of the ω -lemma (if $|\alpha| \leq [s/2]+1$) and from the fact that $D^\alpha g$ is L^2 (if $|\beta| \leq [s/2]+1$).

III. H^S Functions on Manifolds and the Group \mathcal{D}^S .

Let M, N be smooth manifolds, $f: M \rightarrow N$.

Definition 35: For $p \in M$ we say f is H^S at p iff there exist charts (U, ϕ) and $(V, \psi)^*$ about p and $f(p)$ respectively such that $\psi^{-1} \circ (f \upharpoonright \phi(U)) \circ \phi$ is an H^S function. $H^S(M, N)$ is the set of maps from M to N which are H^S at every point of M .

Proposition 36: If a map is H^S at p with respect to one pair of charts (U, ϕ) and (V, ψ) at $f(p)$, then it is H^S with respect to any other pair.

Proof: If (U', ϕ') , (V', ψ') is another pair $(\psi')^{-1} \circ f \circ \phi' = (\psi'^{-1} \circ \psi) \psi^{-1} \circ f \circ \phi \circ (\phi^{-1} \circ \phi')$ on the set $\phi'^{-1}(\phi(U) \cap \phi'(U')) \cap f^{-1}(\psi(V) \cap \psi'(V'))$ which is a neighborhood of p . $\psi'^{-1} \circ \psi$ and $\phi^{-1} \circ \phi'$ are smooth maps whose derivatives are everywhere non-singular. Hence using the composition lemma, $\psi'^{-1} \circ f \circ \phi'$ is H^S if and only if $\psi^{-1} \circ f \circ \phi$ is H^S . q.e.d.

Our next general goal is to define a topology and differentiable structure on $H^S(M, N)$. To do so we must use the following:

* A chart (U, ϕ) at p is a homeomorphism ϕ of an open set in Euclidean space, U , onto a neighborhood of p which defines the differentiable structure on the neighborhood (see 13, p. 16).

Lemma 37: Let U be a bounded set in Euclidean space \mathbb{R}^n . Let $f: U \rightarrow \mathbb{R}$ be an H^s function, and $\lambda \in C_0^\infty(U, \mathbb{R})$. Then $\lambda^2 f \in H_0^s(U)$.

Proof: We first show that λf is an H^s function.

Fix $p \in U$. We know by the definition of H^s functions that there are neighborhoods U_p and V_p of p and

$\rho_p \in C_0^\infty(U_p, \mathbb{R})$ such that $\rho_p \upharpoonright V_p \equiv 1$ and $\rho_p f \in H_0^s(U_p)$.

But $\rho_p \lambda \in C_0^\infty(U_p, \mathbb{R})$ since λ is smooth and $\text{support}(\rho_p) \subseteq U$.

Therefore, $\rho_p \lambda \in H_0^s(U_p, \mathbb{R})$ so $\rho_p \lambda \rho_p f \in H_0^s(U_p)$ by

lemma 29. Hence $\rho_p^2 \lambda f \in H_0^s(U_p)$. Also $\rho_p^2 \upharpoonright V_p \equiv 1$,

so λf is H^s at p . Furthermore,

$\text{support}(\lambda f) = \text{support}(\lambda^2 f) \subseteq \text{support}(\lambda^2) \subseteq U$ so

$\text{support}(\lambda f) = \text{support}(\lambda^2 f)$ is compact.

Now for each $p \in U$, pick U_p , V_p and ρ_p as above, so that $\rho_p \lambda f \in H_0^s(U_p)$. Since $\text{support}(\lambda)$ is compact we can choose a finite number of points $\{p_i\}$ such that

$V = \bigcup_i V_{p_i} \supseteq \text{support}(\lambda)$. Call these V_i , call the

corresponding U_p and ρ_p , U_i and ρ_i .

$\rho_i \lambda f \in H_0^s(U_i) \subseteq H_0^s(U)$ under the natural inclusion (see proposition 4). Therefore $(\sum_i \rho_i) \lambda f \in H_0^s(U)$.

Now consider the function $\frac{\lambda}{\sum_i \rho_i}$. It is defined on

V since, for all $v \in V$, there is some ρ_i such that

$\rho_i(v) = 1$. Also $\text{support}(\lambda) \subseteq V$, so $\frac{\lambda}{\sum_i \rho_i} \in C_0^\infty(V)$.

Therefore we can extend $\frac{\lambda}{\sum \rho_i}$ to an element of $C_0^\infty(U) \subseteq H_0^S(U)$.

Therefore $(\frac{\lambda}{\sum \rho_i})(\sum \rho_i \lambda f) = \lambda^2 f \in H_0^S(U)$. q.e.d.

Now assume M is a compact manifold and fix $f \in H^S(M, N)$.

Let $T_f = \{h \in H^S(M, T(N)) \mid \pi h = f\}$ where π is the projection $\pi: T(N) \rightarrow N$. T_f is a linear space --- the linear structure induced by the linear structure on the fibres of $T(N)$. We shall define an inner product on T_f , making it a Hilbert space.

Let $\{V_j\}$ be a collection of coordinate neighborhoods of N and $\psi_j: T(V_j) \rightarrow V_j \times \mathbb{R}^n$ local trivializations of the bundle $T(N)$. Pick a finite collection of triples $\{U_i, K_i, \rho_i\}$ where $\{U_i\}$ are coordinate neighborhoods of M , K_i is a compact subset of U_i , and $\rho_i \in C^\infty(M, \mathbb{R})$ such that:

- 1) There exists $\lambda_i \in C^\infty(M, \mathbb{R})$ such that $\rho_i = \lambda_i^2$.
- 2) $\text{Support}(\rho_i) \subseteq U_i$ and $\rho_i \upharpoonright K_i \equiv 1$.
- 3) $\bigcup_i \{\text{interior}(K_i)\} = M$.
- 4) For each i , there exists j such that $f(U_i) \subseteq V_j$.

The existence of the collection $\{(U_i, K_i, \rho_i)\}$ follows from the usual arguments used to construct partitions of

unity (see 13, p. 27).

We now define an inner product on T_f .

Let $\pi_2: V_1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by projection on the second factor. Let $h, h' \in T_f$. Let $h_{1j} = \pi_2 \psi_j(\rho_1 h \upharpoonright U_1): U_1 \rightarrow \mathbb{R}^n$ (j is such that $f(U_1) \subseteq V_j$)
 $h'_{1j} = \pi_2 \psi_j(\rho_1 h' \upharpoonright U_1): U_1 \rightarrow \mathbb{R}^n$.

By lemma 37, $h_{1j}, h'_{1j} \in H_0^s(U_1, \mathbb{R}^n)$ so $(h_{1j}, h'_{1j})_s$ is defined.

Definition 38: Let $(h, h') = \sum_1 (h_{1j}, h'_{1j})_s$ be a map from $T_f \times T_f \rightarrow \mathbb{R}$.

It is clear that $(,)$ is a positive definite bilinear map, so it is an inner product on T_f .

Proposition 39: T_f is a Hilbert space with the inner product $(,)$.

Proof: We must show completeness. Assume $\{h^n\}$ is Cauchy in T_f .

Then for all i , $(h_{1j}^n - h_{1j}^m, h_{1j}^n - h_{1j}^m) \rightarrow 0$ as $n, m \rightarrow \infty$.

Therefore, since $H_0^s(U)$ is complete, there exists $h_{1j} \in H_0^s(U)$ such that $h_{1j}^n \rightarrow h_{1j}$ in $H_0^s(U)$. Define $h: M \rightarrow T(N)$, by $h(p) = \psi_j^{-1}(f(p), h_{1j}(p))$ for $p \in \text{interior}(K_1)$.

We must show that this is a consistent definition;
i.e. that if $p \in \text{interior}(K_1) \cap \text{interior}(K_{1'})$,
 $\psi_j^{-1}(f(p), h_{1,j}(p)) = \psi_j^{-1}(f(p), h_{1,j}(p))$, where
 $f(U_1) \subseteq V_j$. To show this we note that for all n
 $\rho_1, \psi_j^{-1}(f(p), h_{1,j}^n(p)) = \rho_1 \psi_j^{-1}(f(p), h_{1,j}^n(p))$, if
 $p \in U_1 \cap U_{1'}$. Support $(\rho_1 \rho_{1'}) \subseteq U_1 \cap U_{1'}$, and $\pi_2 \psi_j h^n$ is
an H^S function on $U_1 \cap U_{1'}$. Therefore by lemma 37
(since $\rho_1 \rho_{1'} = \lambda_1^2 \lambda_{1'}^2$) $\rho_1, h_{1,j}^n = \rho_1 h_{1,j}^n, \in H_0^S(U_1 \cap U_{1'})$,
and $\rho_1, h_{1,j}^n \rightarrow \rho_1, h_{1,j}$ in $H_0^S(U_1 \cap U_{1'})$. By symmetry
 $\rho_1, h_{1,j}^n \rightarrow \rho_1 h_{1,j}$ in $H_0^S(U_1 \cap U_{1'})$. Therefore
 $\rho_1, \psi_j^{-1}(f(p), h_{1,j}(p)) = \rho_1 \psi_j^{-1}(f(p), h_{1,j}(p))$ on $U_1 \cap U_{1'}$. But
 $\rho_1(p) = \rho_{1'}(p) = 1$ on $K_1 \cap K_{1'}$, so h well defined.

It is clear that $\pi h = f$ and since h agrees with
 $p \rightarrow \psi_j^{-1}(f(p), h_{1,j}(p))$ on the open set $\text{interior}(K_1)$,
 $h \in H^S(M, T(N))$. Also $\pi_2 \psi_j(\rho_1 h) \upharpoonright U_1 = h_{1,j}$, so $h_n \rightarrow h$
in T_f and T_f is complete. q.e.d.

Proposition 40: The topology of T_f is independent of
the particular choices of (U_1, K_1, ρ_1) and (V_j, ψ_j) .

Proof: First we recall that to prove two Hilbert space
norms equivalent, one must only show the identity map is
bounded as a map from T_f with one norm to T_f with the
other norm.

Now given any two collections $\{(U_i, K_i, \rho_i), (V_j, \psi_j)\}$ and $\{(U_{i'}, K_{i'}, \rho_{i'}), (V_{j'}, \psi_{j'})\}$ there is a common refinement $\{(U_i \cap U_{i'}, K_i \cap K_{i'}, \rho_i \rho_{i'}), (V_j \cap V_{j'}, \psi_j \upharpoonright (V_j \cap V_{j'}))\}$. We shall show that the norm of the refinement is equivalent to either of the original ones.

First we point out that the norm $(\cdot, \cdot)_s$ on $H^S_0(U_i)$ depends on the chart of U_i . U_i is an open set of M so $H^S_0(U_i)$ is really $H^S_0(\phi^{-1}(U_i))$ where $\phi^{-1}(U_i)$ is an open set of Euclidean space and $\phi: \phi^{-1}(U_i) \rightarrow U_i$ is the chart map. However, given two smooth maps ϕ and ϕ_0 ($\phi_0: \phi_0^{-1}(U_i) \rightarrow U_i$), $\alpha_{\phi^{-1} \circ \phi_0}: H^S_0(\phi_0^{-1}(U_i)) \rightarrow H^S_0(\phi^{-1}(U_i))$ is (by the α -lemma 37) a continuous linear map. It has a continuous inverse, $\alpha_{\phi_0^{-1} \circ \phi}$, so it is a topological linear isomorphism. Hence $H^S_0(U_i)$ is well defined as a "Hilbertable" topological linear space (a space whose topology can be defined by an inner product). Therefore the inner product $(\cdot, \cdot)_s$ on $H^S_0(U_i)$ is defined up to topological equivalence.

Next we remark that for any $f_i \in H^S_0(U_i, \mathbb{R})$, there is a constant C_{f_i} , such that $(f_i h, f_i h)_s \leq C_{f_i} (h, h)_s$. This is a direct consequence of corollary 30 where the bilinear map is scalar multiplication. Hence if in (U_i, K_i, ρ_i) , we replaced ρ_i by ρ_i^2 , the identity map

of T_f with the new norm into T_f with the old norm would be continuous, and therefore an isomorphism.

Now we show that the topology of T_f does not depend on the local trivializations ψ_j on $T(N)$. Let ψ_j and ϕ_j be two such trivializations. Let $\| \cdot \|_1$ be the norm on T_f induced by: $\{(U_i, K_i, \rho_i), (V_j, \psi_j)\}$ and let $\| \cdot \|_2$ be induced by $\{(U_i, K_i, \rho_i^2), (V_j, \phi_j)\}$ which we know is equivalent to that induced by $\{(U_i, K_i, \rho_i), (V_j, \phi_j)\}$. We shall show that $\| \cdot \|_2$ is bounded by a constant times $\| \cdot \|_1$.

Note that $\pi_2 \circ \phi_j \circ \rho_i^{2h} \upharpoonright U_i = \pi_2(\phi_j \circ \psi_j^{-1}) \circ \psi_j \circ \rho_i^{2h} \upharpoonright U_i$. By the definition of a vector bundle, $\phi_j \circ \psi_j^{-1}$ defines a smooth map l_j from V_j to $L(\mathbb{R}^n, \mathbb{R}^n)$ where n is the dimension of the fibre of $T(N)$ (see 13, p. 35).

Also $\pi_2 \circ \phi_j \circ \psi_j^{-1} \circ \psi_j \circ \rho_i^{2h} = (\rho_i(l_j \circ \pi \circ h)) \cdot (\pi_2 \circ \psi_j \circ \rho_i^h)$ where $\pi: T(N) \rightarrow N$ and " \cdot " signifies the action of the linear transformation in $L(\mathbb{R}^n, \mathbb{R}^n)$.

Since l_j is smooth, $l_j \circ \pi \circ h = l_j \circ f$ is H^s and therefore by lemma 37, $\rho_i(l_j \circ \pi \circ h) \in H_0^s(U_i, L(\mathbb{R}^n, \mathbb{R}^n))$. Let $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ be the inner products over U_i giving rise to the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ respectively.

By corollary 30, there is a constant C_1 , depending on $(\rho_i(l_j \circ \pi \circ h \upharpoonright U_i), \rho_i(l_j \circ \pi \circ h \upharpoonright U_i))_s$ such that

$$(h, h)_2 \leq c_1^2 (h, h)_1 .$$

From this it is clear that $\| \cdot \|_2$ is bounded by $\sum c_1 \| \cdot \|_1$, and that $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.

Now we know that the norm (call it $\| \cdot \|_3$) which corresponds to the set $\{(U_1 \cap U_{1'}, K_1 \cap K_{1'}, \rho_1 \rho_{1'}) , (V_j \cap V_{j'}, \psi_j \upharpoonright V_j \cap V_{j'})\}$ is equivalent to the norm $\| \cdot \|_4$ which we define by the same set with ψ_j , replacing ψ_j .

To prove the proposition we need only show that $\| \cdot \|_1$ is equivalent to $\| \cdot \|_3$, for it then follows by symmetry that the norm induced by $\{(U_1, K_1, \rho_1), (V_j, \psi_j)\}$ is equivalent to $\| \cdot \|_4$ and that therefore all are equivalent.

To show this we replace $\rho_1 \rho_{1'}$ by $\rho_1^2 \rho_{1'}^2$ to form a new $\| \cdot \|_3$.

$$\pi_2 \psi_j \rho_1^2 \rho_{1'}^2 h \upharpoonright U_1 \cap U_{1'} = (\rho_1 \rho_{1'})^2 (\pi_2 \circ \psi_j \rho_1 h) \upharpoonright U_1 \cap U_{1'}$$

$$\text{and } \pi_2 \psi_j \rho_1^2 h \upharpoonright U_1 \cap U_{1'} \in H_0^S(U_1 \cap U_{1'}) \subseteq H_0^S(U_1) .$$

$$\text{Also } \rho_1 \rho_{1'}^2 \in H_0^S(U_1 \cap U_{1'}) \subseteq H_0^S(U_1) \text{ and}$$

$$\pi_2 \circ \psi \circ \rho_1 h \upharpoonright U_1 \in H_0^S(U_1) .$$

Therefore, since the above inclusions are continuous by corollary 30, the s-norm of $\pi_2 \circ \psi_j \circ \rho_1^2 \rho_{1'}^2 h$ on U_1 is less than some constant times the product of the s-norms of $\rho_1 \rho_{1'}^2$ and $\pi_2 \psi_j \rho_1 h$ on U_1 . Therefore, there is a

new constant C (depending on ρ_1, ρ_1^2) such that $\| \cdot \|_3$ is bounded by $C \| \cdot \|_1$. The proposition follows.

Remarks: 1) Note that in the construction of T_f we have not used the fact that $T(N)$ is the tangent bundle, but only that it is a vector bundle over N . Hence, if we replace $T(N)$ with any vector bundle E , we get a Hilbert space $T_f \subseteq H^S(M, E)$.

2) If we consider the special case $N = M$ and $f = \text{Identity map}$, then $T_f = \{h \in H^S(M, E) \mid \pi h = \text{Identity}\}$.

T_f , then, is the space of sections of E , which are H^S functions. Palais also defines a Hilbert space of such functions which he calls $H^S(E)$ (see 17, p. 149). We shall show later than our definition is equivalent to his.

Definition 41: Assume f is C^k . Let

$C^k_{T_f} = \{h \in C^k(M, T(N)) \mid \pi h = f\}$. $C^k_{T_f}$ is a linear space with the operations defined the same way as for T_f . Let

$\| \cdot \|$ be a norm on $C^k_{T_f}$ defined by $\|h\| = \sum_{|\alpha| \leq k} \left(\max_{p \in K_1} |D_p^\alpha h_{ij}| \right)$

where K_1 and h_{ij} are defined as in the definition of the Hilbert structure in T_f .

Proposition 42: $C^k_{T_f}$ is a Banach space whose topology is independent of the choices for K_1 and h_{ij} , (as in the T_f situation).

Proof: One can use essentially the same arguments as for T_f (see 1, pp. 14-17).

Proposition 43: $C^1 T_f \supseteq T_f$ and the inclusion is continuous.

Proof: The inclusion is obvious, since we have picked s large enough so that all H^s functions are C^1 . Also, by the Sobolev lemma for each i , there is a constant C_i such that $\|h_{ij}\|_1 \leq C_i \|h_{ij}\|_s$ where $\|\cdot\|_s$ is the norm on $H^s_0(U_i)$ and $\|h_{ij}\|_1 = \max_{p \in U_i} \{|D_p h_{ij}| + |h_{ij}(p)|\}$. But from the definition of h_{ij} it is clear that

$$\|h_{ij}\| \geq \max_{p \in K_i} \{|D_p h_{ij}| + |h_{ij}(p)|\}. \text{ Therefore}$$

$\sum_{\substack{i \\ |\alpha| \leq 1}} \max_{p \in K_i} \{|D^\alpha h_{ij}(p)|\} \leq \sum_i C_i \|h_{ij}\|_s$. The proposition follows.

Now we proceed to define a manifold structure on $H^s(M, N)$ (our method is like that of 6, p. 306). Fix a smooth Riemannian structure on N with exponential map $e: T(N) \rightarrow N$. We know that e is defined on a neighborhood of the zero section of $T(N)$. Also we can pick some smaller neighborhoods θ, θ' with $\bar{\theta}$ the closure of θ in θ' , such that $e|_{\theta'}$ is a diffeomorphism. $e|_{\theta'}$ restricted to a fibre $T_p(N)$ defines a normal coordinate system at p (see 15, p. 59). But $f: M \rightarrow N$ is continuous, so $f(M)$ is a compact subset of N . Hence there

is an ε such that $\Theta \cap \pi^{-1}(f(M))$ contains all vectors in $\pi^{-1}(f(M))$ of length less than ε .

Now let $\Theta_f = \{h \in T_f \mid h(M) \subseteq \Theta\}$ and define $\psi_f: \Theta_f \rightarrow H^S(M, N)$ by $\psi_f(h) = e \cdot h$. It is clear that $e \cdot h$ is H^S , since h is H^S and e is smooth.

Lemma 44: Θ_f is a neighborhood of zero in T_f and $\psi_f \upharpoonright \Theta_f$ is injective.

Proof: First we show that ψ_f is injective. Take $h, h' \in \Theta_f$ and assume $eh = eh'$. Then for all $p \in M$, $eh(p) = eh'(p)$ and $h(p), h'(p) \in \Theta \cap \pi^{-1}(p)$. Therefore, $h(p) = h'(p)$ since e is injective on the fibres of Θ . Hence $h = h'$. Now we show that Θ_f is a neighborhood of zero in T_f . Recall $T_f \subseteq C^0 T_f$ and the injection is continuous. Therefore if B_δ is a δ -ball about 0 in $C^0 T_f$, $B_\delta \cap T_f$ is a neighborhood of 0 in T_f . We shall show that $\Theta_f \supseteq B_\delta \cap T_f$ for some δ .

For each coordinate neighborhood V_j in N the Riemannian metric on N is represented by functions $g_{kl}: V_j \rightarrow \mathbb{R}$, where $(g_{kl}(p))$ is a symmetric positive definite matrix for all $p \in V_j$. Let ψ_j be the usual trivialization induced by the coordinate system

$$(\psi_j(\sum_1 a_i \frac{\partial}{\partial x_i}(p)) = (p, (a_1, a_2, \dots, a_n))) .$$

Let $g_{kl}^{1/2}$ be the unique positive definite symmetric matrix such that $(g_{kl}^{1/2})^2 = g_{kl}$, and let $g_{kl}^{-1/2}$ be its inverse. Then if $X \in T_p(N)$, $(g_p^{-1/2}X, g_p^{-1/2}X)_p = |\psi_j(X)|^2$. $(g_{kl}^{-1/2})_p$ is an element of $L(\mathbb{R}^n, \mathbb{R}^n)$ with inverse $(g_{kl}^{1/2})$, and the map $p \rightarrow g_p^{1/2}$ is continuous. Therefore, there is a δ_j such that if $|V| < \delta_1$, $|g_p^{1/2}V| < \varepsilon$ for all $p \in K_1$. Assume $X \in \pi^{-1}(V_j)$ such that $|\psi_j(g^{1/2}X)| < \varepsilon$. Then $(X, X)^{1/2} < \varepsilon$. Therefore if $h \in C^0 T_f$ such that $\max_{p \in K_1} \{|h_{1j}(p)|\} < \delta_j$, then for all $p \in K_1$, $(h(p), h(p)) < \varepsilon$. Let $\delta = \min \{\delta_j\}$ (the minimum of a finite set, since there are only a finite number of K_i).

Now clearly $\Theta_f \supseteq B_\delta \cap T_f$ since B_δ consists only of elements $h \in C^0 T_f$ such that for any p in M , $(h(p), h(p))^{1/2} < \varepsilon$. The lemma follows.

We define $H^S(M, N)$ to be a manifold where the set $\bigcup_{f \in H^S(M, N)} (\Theta_f, \psi_f)$ is a collection of charts for $H^S(M, N)$. We shall call these standard charts. To be sure that this definition makes sense, we must check that if $f, f' \in H^S(M, N)$ $\psi_{f'}^{-1} \circ \psi_f : \psi_f^{-1}(\psi_f(\Theta_f) \cap \psi_{f'}(\Theta_{f'}))$ is a smooth map. (Then by symmetry of f and f' we can say that it has smooth inverse as well.)

Proposition 45: $\psi_f^{-1} \circ \psi_f$ is smooth on $\psi_f^{-1}(\psi_f(\theta_f) \cap \psi_{f'}(\theta_{f'}))$.

Proof: Let $E: \theta \subseteq T(N) \rightarrow N \times N$ by $E(X) = (\pi(x), e(x))$. Then $(\psi_f^{-1} \circ \psi_f(h))(p) = E^{-1}(f'(p), eh(p))$. We note that θ has been chosen so that E and E^{-1} are smooth maps in a neighborhood of $\bar{\theta}$. Hence E and E^{-1} have bounded derivatives on θ and $E(\theta)$ respectively, and e also has bounded derivatives on θ .

The remainder of the proof falls in the context of a general procedure for determining smoothness of maps on spaces such as T_f . Therefore we now discuss this procedure, which is essentially a globalization of the ω -lemma.

Lemma 46: Let $\{L_i \subseteq W_i \subseteq K_i \subseteq U_i\}$ be a collection of sets in M such that there is a collection of pairs (V_j, ψ_j) on N and functions $\{\rho_i\}$, $\{\sigma_i\}$ on M so that $\{(U_i, K_i, \rho_i), (V_j, \psi_j)\}$ and $\{(W_i, L_i, \sigma_i), (V_j, \psi_j)\}$ define norms, $\|\cdot\|_1$, and $\|\cdot\|_2$ on T_f . Define a new inner product $(\cdot, \cdot)_3$ on T_f by $(h, h')_3 = \sum_i \langle h_{ij} \upharpoonright W_i, h'_{ij} \upharpoonright W_i \rangle_s$, where $h_{ij} = \rho_i(\pi_2 \psi_j h \upharpoonright U_i)$, $h'_{ij} = \rho_i(\pi_2 \psi_j h' \upharpoonright U_i)$ and $\langle \cdot \rangle_s$ is the inner product on $H^S(W_i)$.

Then the norm $\|\cdot\|_3$ on T_f defined by $(\cdot, \cdot)_3$ is equivalent to $\|\cdot\|_1$ and $\|\cdot\|_2$.

Proof of the Equivalence: We know that $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent, so it is enough to find a constant $C > 0$, such that $\| \cdot \|_1 \geq \| \cdot \|_3 \geq C \| \cdot \|_2$. $\| \cdot \|_1 \geq \| \cdot \|_3$ because $\langle h_{ij}, h'_{ij} \rangle_s \geq \langle h_{ij} \upharpoonright W_i, h'_{ij} \upharpoonright W_i \rangle_s$ also by Corollary 31, we know that there is a constant C_1 such that $\langle \sigma_i h_{ij} \upharpoonright W_i, \sigma_i h'_{ij} \upharpoonright W_i \rangle_s \leq C_1 \langle h_{ij} \upharpoonright W_i, h'_{ij} \upharpoonright W_i \rangle_s$. The lemma follows if we let $C = \min_i \{ \frac{1}{C_1} \}$.

Procedure 47: Let $\{W_i\}$ and $\{W'_i\}$ be sets of the above form defining norms for T_f and T'_f respectively. For $h \in T_f$ we denote $\pi_2 \circ \psi_j \circ h \upharpoonright W_i$ by h_i , and for $h' \in T'_f$, we denote $\pi_2 \circ \psi_j \circ h' \upharpoonright W'_i$ by h'_i .

Let Q be an open set in T_f and let $\phi: Q \rightarrow T'_f$. Consider the map $h_i \rightarrow \phi(h)_i$ from $Q_i = \{h_i | h \in Q\}$ into $H^S(W'_i)$, where Q_i has the subset topology of $H^S(W_i)$. ϕ is C^0 if $(h^n)_i \rightarrow h_i$ in Q_i implies $\phi(h^n)_i \rightarrow \phi(h)_i$ in $H^S(W'_i)$.

Assume ϕ is C^{k-1} . To check that ϕ is C^k we do the following: Make sure that for $h \in Q$, $v^1 \dots v^k \in T_f$, $d^k \phi(h_i, v^1_i \dots v^k_i)$ exists for all i . Then by definition of the norm on T'_f , we know that $d^k \phi(h, v^1 \dots v^k)$ exists. Check that $d^k \phi(h, v^1 \dots v^k)$ defines a continuous linear map from Q to $L^k(T_f, T'_f)$. Then by corollary 15, ϕ is C^k .

Example 48: Let $g: T(N) \rightarrow T(N)$ be a smooth map which takes fibres into fibres so that $\omega_g: T_f \rightarrow T_f$, by $h \rightarrow g \circ h$. Then ω_g is smooth.

Proof: To show $h_i \rightarrow \omega_g(h)_i$ continuous and to get the existence of $d\omega_g(h,v)$ we use steps one and two of the ω -lemma. The same estimates work because for any $p \in W_1$, $\omega_g(h)_i(p) = \pi_2 \circ \psi_j \circ g \circ \psi_j^{-1}(p, h_i(p))$. Therefore $d\omega_g(h,v)(p) = D_{(p,h_i(p))}(\pi_2 \circ \psi_j \circ g \circ \psi_j^{-1})(0, v_i(p))$.

Hence $d\omega_g(h,v) = D_h g(v)$, so $D_h \omega_g(v) = D_h(g)(v)$. Assuming ω_g , C^{k-1} and $D_h^{k-1} \omega_g(v^1 \dots v^{k-1}) = D_h^{k-1} g(v^1 \dots v^{k-1})$, we note, using the ω -lemma (step 6), that $d^k \omega_g(v^1 \dots v^k) = d \omega_{D^{k-1} g(v^1 \dots v^{k-1})}(h, v^k)$ which exists and equals $D_h^k g(v^1 \dots v^k)$. Therefore, by corollary 15, ω_g is C^k .

Remark: We cannot say that the map induced by ω_g from $H^S(W_1)$ to $H^S(W'_1)$ is smooth, because the estimates of the ω -lemma require that v^1 be bounded in the C^0 sense (to get existence of $d\omega_g(h^1, v^1)$). This is not true for any element of $H^S(W_1)$, but it is true for v^1 if $v^1 = \pi \circ \psi_j \circ v \mid W_1$, $v \in T_f$.

Now it is easy to show the smoothness of $\psi_f^{-1} \circ \psi_f$. If h is in the domain, $h_i \rightarrow (\psi_f^{-1} \circ \psi_f)(h)_i$ is the

composition $h_1 = \pi_2 \psi_j h \rightarrow (f, \pi_2 \psi_j h) \rightarrow e \psi_j^{-1}(f, \pi_2 \psi_j h) = eh \rightarrow (f', eh) \rightarrow \pi_2 \psi_j' E^{-1}(f', eh)$. The maps π_2, ψ_j, e, E^{-1} and ψ_j' are smooth, so composing with them is a smooth operation. Also, the maps $h_1 \rightarrow (f, h_1)$ and $eh \rightarrow (f', eh)$ are clearly smooth. Since the composition of Gateaux derivatives is the derivative of the composed maps (see lemma 14) all Gateaux derivatives exist in $H^S(W_1')$ (we can let $W_1' = W_1$ here). Also

$$d(\psi_{f'}^{-1} \circ \psi_f)(h, v) = D_{(f', eh)}(0, D_h e(v)) \text{ and}$$

the corresponding formula holds for higher derivatives. Hence, the procedure 47 applies, so $\psi_{f'}^{-1} \circ \psi_f$ is smooth. This proves proposition 45.

Now we know that $H^S(M, N)$ is a smooth manifold such that for any $f \in H^S(M, N)$, a small neighborhood of f looks like (i.e., is homeomorphic to) a neighborhood of 0 in T_f . We restrict to the case $M = N$ and define $\mathcal{D}^S = \{f \in H^S(M, M) | f \text{ bijective and } f^{-1} \in H^S(M, M)\}$. We shall show that this set is a group under the operation of composition of mappings, and that it is an open subset of $H^S(M, M)$, and therefore a manifold.

Lemma 49: If $f \in \mathcal{D}^S$, then f is C^1 and for all $p \in M$, $T_p f$ is a linear isomorphism from $T_p(M)$ to $T_{f(p)}(M)$.

Proof: f and f^{-1} are C^1 by the remark following the Sobolev lemma. Therefore $Tf: T(M) \rightarrow T(M)$ and $Tf^{-1}: T(M) \rightarrow T(M)$ are well defined functions and are inverse to each other. Hence, for any $p \in M$, $T_p f$ and $T_{f(p)} f^{-1}$ are linear maps which are inverse to each other, so they are both isomorphisms. q.e.d.

Remark: This tells us that all elements of \mathcal{D}^s are C^1 homeomorphisms with C^1 inverse.

Corollary 50: \mathcal{D}^s is closed under composition of mappings.

Proof: We shall show $f, g \in \mathcal{D}^s$ implies $f \circ g \in \mathcal{D}^s$, by passing to the local situation and using the composition lemma. Fix $p \in M$. Pick coordinate neighborhoods U, V, W of $p, g(p)$ and $f \circ g(p)$, respectively, so that $g(U) \subseteq V$, $f(V) \subseteq W$. Then there is a neighborhood U_0 of p such that $U_0 \subseteq U$ and $g \in H^s(U_0)$. $g(U_0)$ is a neighborhood of $f(p)$ so there are neighborhoods V_0, V_1 of $f(p)$ and $\rho \in C_0^\infty(V_1, \mathbb{R})$ such that $V_0 \subseteq V_1 \subseteq g(U_0)$, $\rho f \in H_0^s(V_1)$ and $\rho \upharpoonright V_0 \equiv 1$. $g \upharpoonright g^{-1}(V_1) \in H^s(g^{-1}(V_1))$ since $g^{-1}(V_1) \subseteq U_0$. Therefore, by the composition lemma $(\rho f) \circ g \upharpoonright g^{-1}(V_1) \in H^s(g^{-1}(V_1))$, so $(\rho f) \circ g \upharpoonright g^{-1}(V_0) \in H^s(g^{-1}(V_0))$ but $\rho f = f$ on V_0 , so $f \circ g \upharpoonright g^{-1}(V_0) \in H^s(g^{-1}(V_0))$.

Hence $f \circ g$ is H^s at p , so $f \circ g$ is H^s . Similarly $g^{-1} \circ f^{-1}$ is H^s , so $(f \circ g)^{-1}$ is H^s , and $f \circ g \in \mathcal{D}^s$.
q.e.d.

Remark: This tells us that \mathcal{D}^s is a group under the operation of composition of mappings.

Lemma 51: If f is a bijective map in $H^s(M, M)$, and f^{-1} is a C^1 map, then $f^{-1} \in H^s(M, M)$ so $f \in \mathcal{D}^s$.

Proof: We assume s large enough so that $H^{[s/4]}$ functions on M are C^1 . Then H^s functions will be $C^{[s/2]+1}$, and $H^{([s/2]+1)/2}$ functions will be C^1 . Now use induction. f is a $C^{[s/2]+1}$ map with C^1 inverse. Therefore, f^{-1} is also $C^{[s/2]+1}$ (see 13, p. 13). Therefore, f^{-1} is $H^{[s/2]+1}$ and $[s/2]+1$ is large enough so that the statements of the composition, α - and ω -lemmas hold. Assume f^{-1} is H^k for $[s/2]+1 \leq k \leq s-1$. Consider f locally with range and domain in a Euclidean space. We know that $D_{f(p)}(f^{-1}) = (Df_p)^{-1}$. Therefore, $D(f^{-1})$ is the map $q \rightarrow f^{-1}(q) \rightarrow (D_{f^{-1}(q)} f) \rightarrow (D_{f^{-1}(q)} f)^{-1}$, the composition of f^{-1} , Df and matrix inverse (call it i). But i is smooth and Df is H^{s-1} . Therefore, by remark 3 following the composition lemma 28, $i \circ Df$ is H^{s-1} . But f^{-1} is H^k and $D(f^{-1})$ is everywhere non-singular. Hence, by the composition lemma, $i \circ Df \circ f^{-1}$

is H^k . Therefore $D(f^{-1})$ is H^k , so f^{-1} is H^{k+1} .
The lemma follows.

Proposition 52: \mathcal{D}^S is an open subset of $H^S(M, M)$.

Proof: By the above lemma, we can consider \mathcal{D}^S as the set of bijective H^S maps with C^1 inverses. Pick $f \in \mathcal{D}^S$ and $\{f_n\} \subseteq H^S(M, M)$ such that $f_n \rightarrow f$ in $H^S(M, M)$. Then for sufficiently large n , we can find $h_n \in T_f$, such that $\psi_f(h_n) = f_n$. Since $f_n \rightarrow f$, $h_n \rightarrow 0$ in T_f . Therefore, by proposition 43, $h_n \rightarrow 0$ in $C^1 T_f$.

Look at f_n and h_n locally:

$f_n = e h_n$, so $Df_n = D_{h_n} e \cdot Dh_n$. e is a C^∞ function with bounded derivative (on $\theta \subseteq T(M)$). So, as in the ω -lemma, $\{Dh_n\}$ converges implies Df_n converges. Therefore, $f_n \rightarrow f$ in the C^1 sense: That is, for all $p \in M$, $f_n(p) \rightarrow f(p)$ and $D_p f_n \rightarrow D_p f$, and this convergence is uniform in p .

It is well known (see 16, p. 21) that if f is a C^1 map with C^1 inverse, so is any map sufficiently near it, in the C^1 sense. Hence for large n , f_n is a C^1 map with C^1 inverse, so $f_n \in \mathcal{D}^S$. Therefore, \mathcal{D}^S is open in $H^S(M, M)$. q.e.d.

Remark: We now have a manifold structure on \mathcal{D}^S , inherited from the structure on $H^S(M, M)$.

\mathcal{D}^S is a smooth manifold and a group; the question arises whether the group operation is smooth. This we shall not prove --- it seems that it is false (7) --- but we will obtain some lesser results in this direction.

Proposition 53: Let $\zeta \in \mathcal{D}^S$ and define $R_\zeta: \mathcal{D}^S \rightarrow \mathcal{D}^S$ to be the map $f \rightarrow f \circ \zeta$. Then R_ζ is smooth.

Proof: Fix $f \in \mathcal{D}^S$ and let (ψ_f, U_f) $(\psi_{f\zeta}, U_{f\zeta})$ be standard charts of \mathcal{D}^S at f and $f\zeta$ where U_f and $U_{f\zeta}$ are open neighborhoods of 0 in T_f and $T_{f\zeta}$ respectively.

Then $\psi_{f\zeta}^{-1} \circ R_\zeta \circ \psi_f$ is the map $h \rightarrow e \circ h \rightarrow e \circ h \circ \zeta \rightarrow E^{-1}(f \circ \zeta, e \circ h \circ \zeta) = h \circ \zeta$. We must show $\psi_{f\zeta}^{-1} \circ R_\zeta \circ \psi_f$ to be smooth. The map $A_\zeta: h \rightarrow h \circ \zeta$ is a linear map of T_f to $T_{f\zeta}$. Therefore, we need only show that it is continuous. For this we shall use the α -lemma.

Pick coordinate neighborhoods $\{U_i\}$ covering M , such that $\{\zeta^{-1}(U_i)\}$ are also coordinate neighborhoods, and such that there are sets $L_i \subseteq W_i \subseteq K_i \subseteq U_i$, and functions ρ_i, σ_i , and pairs (V_j, ψ_j) , which satisfy the properties of lemma 46, and also the following property: For each i , there are functions $\rho_i' \in C_0^\infty(\zeta^{-1}(U_i))$ and $\sigma_i' \in C_0^\infty(\zeta^{-1}(W_i))$ so that $\{(\zeta^{-1}(U_i), \zeta^{-1}(K_i), \rho_i'), (V_j, \psi_j)\}$

and $\{(\zeta^{-1}(W_1), \zeta^{-1}(L_1), \sigma'_1), (V_j, \psi_j))\}$ define norms on $T_f \zeta$. (For the construction of ρ'_1, σ'_1 see 10, p. 3.)

Now we get norms $\| \cdot \|_1, \| \cdot \|_2$ and $\| \cdot \|_3$ on T_f and $\| \cdot \|'_1, \| \cdot \|'_2$ and $\| \cdot \|'_3$ on $T_f \zeta$ as in lemma 46.

We shall show that $A_\zeta: T_f \rightarrow R_{f \circ \zeta}$ is bounded in the norms $\| \cdot \|_1$ and $\| \cdot \|'_3$ on T_f and $T_f \zeta$. By the definition of these norms, it is enough to show that for each i , the map A_ζ^i defined by $h_{ij} \rightarrow h_{ij} \circ \zeta \upharpoonright \zeta^{-1}(W)$ is bounded in the norms of $H^s_0(U_1)$ and $H^s(\zeta^{-1}(W_1))$. But this map is the composition

$H^s_0(U_1) \xrightarrow{A} H(\zeta^{-1}(U_1)) \xrightarrow{B} H^s(\zeta^{-1}(W_1))$ where A is the map $\alpha(\zeta \upharpoonright \zeta^{-1}(U_1))$ and B is the restriction map.

$(B(f) = f \upharpoonright \zeta^{-1}(W_1))$. A is continuous linear by the α -lemma, and B is because it is norm decreasing. The proposition follows:

Proposition 54: Let $\zeta \in \mathcal{D}$. (i.e. $\zeta \in \mathcal{D}^S$ and ζ is a smooth map).

Define $L_\zeta: \mathcal{D}^S \rightarrow \mathcal{D}^S$ by $L_\zeta(f) = \zeta \circ f$. Then L_ζ is a smooth map.

Proof: Fix $f \in \mathcal{D}^S$. We look at the map

$Z_\zeta = \psi_{\zeta \circ f}^{-1} \circ L_\zeta \circ \psi_f$ which we claim is defined near the origin of T_f and into $T_{\zeta f}$. From the definition of

$\theta_{\zeta f}$, we know that $h \in \theta_{\zeta f}$ if and only if there is an $\varepsilon > 0$

such that for all $p \in M$, $\langle h(p), h(p) \rangle < \varepsilon$. Since $\theta_{\zeta f}$ was defined using normal coordinates, we know that for any such h , the distance from p to $eh(p)$ is equal to the length of $h(p)$.

Since M is compact and ζ is continuous, there is a δ such that if the distance from p to q in M is less than δ , then the distance between $\zeta(p)$ and $\zeta(q)$ is less than ε .

Restrict θ_f so that it contains only vectors of length less than δ . We claim Z_ζ is defined on the restricted θ_f . Take $h \in \theta_f$. Then by the above, for any p in M , the distance between $\zeta(p)$ and $\zeta eh(p)$ is less than ε . But $L_\zeta \circ \psi_f(h) = \zeta eh$. Therefore, $\psi_{\zeta f}^{-1} \circ L_\zeta \circ \psi_f(h)$ is defined and it equals $E^{-1}(\zeta f, \zeta eh)$.

To conclude the proof, we need only show that $Z_f: \theta_f \rightarrow T_{\zeta f}$ is smooth. This of course, follows from the general procedure as in the case of $\psi_f^{-1} \circ \psi_f$, because we know that E^{-1} , ζ and e are smooth maps with bounded derivatives.

Corollary 55: For any $\zeta \in \mathcal{D}^S$, $L_\zeta: \mathcal{D}^S \rightarrow \mathcal{D}^S$ is continuous.

Proof: $Z_\zeta: \theta_f \rightarrow T_{\zeta f}$ as above. We recall the procedure for checking continuity. $h_1 \rightarrow Z_\zeta(L)_1$ is the composition

$$h_1 \xrightarrow{1} (f, h_1) \xrightarrow{2} e\psi_j^{-1}(h_1) = eh \xrightarrow{3} (\zeta f, \zeta eh) \\ \xrightarrow{4} \pi\psi_j, E^{-1}(\zeta f, \zeta eh) .$$

Here all steps but 3 are clearly continuous. But $T_p\zeta$ is everywhere non-singular and $h_1^n \rightarrow h_1$ in $H^S(W_1)$ means $h_{1j}^n \rightarrow h_{1j}$ in $H_0^S(U_1)$ so $h_1^n \rightarrow h_1$ in $H^{[s/2]+1}(W_1)$. Therefore, by the weak ω -lemma 3 is continuous, so the sequence is also. This proves the corollary.

Proposition 56: $i: \mathcal{D}^S \rightarrow \mathcal{D}^S$ by $\eta \rightarrow \eta^{-1}$ is continuous, so \mathcal{D}^S is a topological group.

Proof: Since R_η and L_η are continuous, we need only show that i is continuous at Id . But in the chart Θ_{Id} , $I = \psi_{\text{Id}}^{-1} \circ i \circ \psi_{\text{Id}}$ has the following form: $I(h)$ is the map $p \rightarrow E^{-1}(eh(p), p)$. This map is smooth by the usual reasoning. The proposition follows.

This concludes our discussion of H^S functions and of \mathcal{D}^S . We shall now look at \mathcal{M}^S , the space of H^S Riemannian metrics on M .

IV. The Space of H^S Metrics.

Let $S^t T^*$ be the bundle of symmetric co-variant t -tensors over M (the symmetrization of the cotangent bundle tensored with itself t times). Let $H^S(S^t T^*)$ be the Hilbert space of H^S sections of $S^t T^*$ (defined as T_f where $f = \text{Id}: M \rightarrow M$ and $T(N)$ is replaced by $S^t T^*$). Let $C^k(S^t T^*)$ be the Banach space of C^k sections of $S^t T^*$ (defined as $C^k_{T_f}$).

Definition 57: The H^S , respectively C^k , respectively smooth Riemannian metrics are the H^S , respectively C^k , respectively smooth, sections γ of $S^2 T^*$ such that for all $p \in M$, $\gamma(p)$ defines a positive definite bilinear form on $T_p(M)$.

We shall call the set of H^S metrics \mathcal{M}^S , the set of C^k metrics $C^k \mathcal{M}$ and the set of smooth metrics \mathcal{M} .

Lemma 58: $C^0 \mathcal{M}$ is an open subset of $C^0(S^2 T^*)$.

Proof: We use the fact that the set of positive definite symmetric matrices is open in the set of symmetric matrices.

Let $\{U_i\}$ be a finite set of coordinate neighborhoods of M , $\{K_i\}$ a collection of compact sets such that $K_i \subseteq U_i$ and $\bigcup K_i = M$. If $\psi_i: S^2 T^*(U_i) \rightarrow U_i \times \mathbb{R}^m$ is the natural trivialization induced by a coordinate system

$x^1 \dots x^n$; that is, $\psi_1((\sum_{jk} g_{jk} dx^j dx^k)_p) = (p, g_{11}, g_{12}, \dots, g_{nn})$,
 then $(g_{jk})_p$ is a positive definite matrix for all $p \in U_1$.
 g_{jk} is a continuous function on U_1 and K_1 is compact.

Therefore, because of the fact that the set of positive definite matrices is open, we can find an ε_1 , such that for any section h of S^2T^* if

$\max_{p \in K_1} \{ \|\pi_2 \circ \psi_1(h) - \pi_2 \circ \psi_1(g)\| \} < \varepsilon_1$, then

$\psi_1(h)_p = (h_{jk})_p$ is positive definite for all $p \in K_1$.

Now clearly if $\varepsilon = \min_1 \{\varepsilon_1\}$ and $C^0 S^2T^*$ has the norm

$\|\gamma\| = \sum_1 \max_{p \in K_1} \{ |\pi_2 \circ \psi_1(\gamma_p)| \}$ then $\|h - \gamma\| < \varepsilon$ implies

h everywhere positive definite, or $h \in C^0 \mathfrak{M}$. q.e.d.

Lemma 59: $C^k \mathfrak{M} (\mathfrak{M}^s)$ is a positive cone in $C^k(S^2T^*)$,
 $(H^s(S^2T^*))$.

Proof: If A and B are positive definite symmetric matrices and a, b are positive real numbers, $aA + bB$ is a positive definite symmetric matrix. The lemma follows directly.

Proposition 60: For all k , $C^k \mathfrak{M}$ is an open positive cone in $C^k(S^2T^*)$. Also for sufficiently large s , \mathfrak{M}^s is an open positive cone in $H^s(S^2T^*)$.

Proof: The injections $C^k(S^2T^*) \subseteq C^0(S^2T^*)$ and $H^S(S^2T^*) \subseteq C^0(S^2T^*)$ are continuous. $C^k \mathcal{M} = C^0 \mathcal{M} \cap C^k(S^2T^*)$ and $\mathcal{M}^S = C^0 \mathcal{M} \cap H^S(S^2T^*)$. The proposition follows.

Since \mathcal{M}^S is an open set in $H^S(S^2T^*)$, it has a natural manifold structure based on $H^S(S^2T^*)$, and for each $\gamma \in \mathcal{M}^S$, $T_\gamma(\mathcal{M}^S)$ is naturally identified with $H^S(S^2T^*)$; that is, $T(\mathcal{M}^S) = H^S(S^2T^*) \times \mathcal{M}^S$.

Definition 61: Let X be a manifold based on a Hilbert space. Let $B(T(X))$ be the bundle of bounded symmetric bilinear forms of $T(X)$, (see 19, p. 155). A Riemannian metric on X is a section γ of $B(T(X))$ such that for all $p \in X$, the bilinear form γ_p on $T_p(X)$ is positive definite and defines the given topology of $T_p(X)$.

Let $B(H^S(S^2T^*))$ be the set of symmetric bounded bilinear forms on $H^S(S^2T^*)$. (It is a Banach space under the operator norm $\|b\| = \max_{\|x\|=\|y\|=1} \{|b(x,y)|\}$.)

Since $T(\mathcal{M}^S) = \mathcal{M}^S \times H^S(S^2T^*)$, $B(T(\mathcal{M}^S)) = \mathcal{M}^S \times B(H^S(S^2T^*))$. Hence a Riemannian metric on \mathcal{M}^S is a smooth map $\mu: \mathcal{M}^S \rightarrow B(H^S(S^2T^*))$ such that for all $\gamma \in \mathcal{M}^S$, μ_γ is a positive definite bilinear form on $H^S(S^2T^*) (\cong T_\gamma(\mathcal{M}^S))$ which induces the topology on $H^S(S^2T^*)$. We shall construct a metric on \mathcal{M}^S which is invariant under the action of \mathcal{D} on \mathcal{M}^S . To define this metric we use the

definition of $H^S(S^2T^*)$ given in 17, p. 149, so we must first discuss that definition and show that it coincides with ours.

We briefly review the construction of $H^S(S^2T^*)$ in (17).

Let E be a vector bundle over M , and $C^\infty(E)$ be the smooth sections of E . Let $C^\infty(M, \mathbb{R})$ be the ring of the smooth real-valued functions on M . Fix $p \in M$ and let I_p be the ideal of functions in $C^\infty(M, \mathbb{R})$ which vanish at p . $C^\infty(E)$ is a module over $C^\infty(M, \mathbb{R})$. Let $Z_p^k(E) = I_p^{k+1} \cdot C^\infty(E)$ and let $J_p^k(E) = C^\infty(E)/Z_p^k(E)$. $J_p^k(E)$ is called the set of k -jets of E at p . We shall define a vector bundle $J^k(E)$, whose fibre at each $p \in M$ is $J_p^k(E)$.

Let U be a coordinate neighborhood of M containing p , such that E is trivial over U , and let $x^1 \dots x^n$ be coordinate functions on U . Let $s^1 \dots s^m$ be smooth sections of E over U which are a basis for the fibre of U at each point. Consider the set of sections of E of the form $\sum_i p_i(x^1 \dots x^n) s^i$ where $p_i(x^1 \dots x^n)$ are polynomials in $\{x^1 \dots x^n\}$. It is a linear space. We know that any section of E over U has the form $\sum_i f_i s^i$, where f_i are smooth functions on U . By Taylor's theorem

we know that at fixed p in U for each f_i , there exists some polynomial p_i such that $f_i - p_i \in I_p^{k+1}$. Also, if a polynomial p_i is in I_p^{k+1} , its first $k+1$ derivatives are zero, so it must be zero. Hence $J_p(E)$ is isomorphic to the vector space whose elements have form $\sum p_i s^i$.

To define the vector bundle $J^k(E)$ we need only specify sufficiently many smooth local sections; i.e. enough smooth local sections to span each fibre $J_p(E)$. The elementary functions $1, x^1, x^2, \dots, x^n, (x^1)^2, x^1 x^2, \dots, (x^n)^k$ span the set of polynomials of degree less than or equal to k . Hence the elementary functions times s^1, s^2, \dots, s^n span $J_p^k(E)$. Let these be smooth sections of $J^k(E)$ over U . It is easy to see that these functions transform smoothly on overlapping coordinate systems, so they define a smooth bundle $J^k(E)$.

There is a natural linear injection $i_k: C^\infty(E) \rightarrow C^\infty(J^k(E))$. If $s \in C^\infty(E)$, $i_k(s)_p$ is defined to be the element represented by s in $C^\infty(E)/Z_p^k(E)$. It follows from the definition of $J^k(E)$ that $i_k(s)$ is smooth whenever s is.

Palais defines $H^S(E)$ as follows: Let $J_k(E)$ have a smooth Riemannian structure --- a positive definite inner product $\langle \cdot, \cdot \rangle$ on each fibre such that the product of any two smooth sections is a smooth function on M .

Define $H^0(J^k(E))$ to be the completion of $C^\infty(J^k(E))$ with respect to the norm induced from the inner product $(s, s') = \int_M \langle s_p, s'_p \rangle d\rho(p)$ where ρ is a smooth measure on M . Palais shows that $H^0(J^k(E))$ as a topological vector space does not depend on the Riemannian structure for $J^k(E)$ or on the measure ρ .

$j_k: C^\infty(E) \rightarrow C^\infty(J^k(E)) \subseteq H^0(J^k(E))$ is a linear injection. Therefore, an inner product on $H^0(J^k(E))$ is an inner product on $C^\infty(E)$. Define $H^S(E)$ to be the completion of $C^\infty(E)$ with respect to this inner product.

Now our proof of the equivalence of the two definitions will be done with two lemmas.

A-Lemma 62: $C^\infty(E)$ is dense in $\overset{\circ}{H}^S(E)$ where $\overset{\circ}{H}^S(E)$ is the set of H^S sections of E with our topology.

B-Lemma 63: The identity map on $C^\infty(E)$ is continuous as a map from $C^\infty(E)$ as a subset of $H^S(E)$ (Palais topology) to $C^\infty(E)$ as a subset of $\overset{\circ}{H}^S(E)$.

Proof of A-Lemma: Pick $\{(U_i, K_i, \rho_i), (V_j, \psi_j)\}$ to define a norm $\| \cdot \|$ for $\overset{\circ}{H}^S(E)$ and let $\{\sigma_i\}$ be a smooth partition of unity subordinate to $\{U_i\}$ such that for each i , there is a smooth function λ_i such that $\sigma_i = \lambda_i^2$. (This can be done, for given any partition of unity σ'_i , let $\sigma_i = (\sigma'_i)^2 / \sum (\sigma'_i)^2$. Then $\lambda_i = \sigma'_i / \sqrt{\sum (\sigma'_i)^2}$ which

is clearly smooth since $\sqrt{\sum(\sigma_i)^2}$ is nowhere zero.)

Now pick $h \in H^S(E)$. By lemma 37, $\pi_2 \circ \psi_j \circ (\sigma_1 h \upharpoonright U_1)$
 $= \sigma_1(\pi_2 \circ \psi_j \circ h \upharpoonright U_1) \in H_0^S(U_1)$. Pick $h_1^0 \in C_0^\infty(U_1)$ such
 that $\|h_1^0 - \sigma_1(\pi_2 \circ \psi_j \circ h \upharpoonright U_1)\|_1 < \varepsilon$ where $\|\cdot\|_1$ is
 the norm on $H_0^S(U_1)$. Define

$$h_1^\infty: M \rightarrow E \text{ by } h_1^\infty(p) = \begin{cases} \psi_j^{-1}(p, h_1^0(p)) & p \in U_1 \\ \psi_j^{-1}(p, 0) & p \notin U_1 \end{cases}$$

h_1^∞ is in $C^\infty(E)$. Let $h^\infty = \sum h_1^\infty$. We shall show that
 h^∞ is close to h in $H^S(E)$. We know $h = \sum \sigma_1 h_1$, so
 it is sufficient to show that for each i , h_1^∞ is close
 to $\sigma_1 h$. $\|h_1^\infty - \sigma_1 h\| = \sum_k \|\rho_k \pi_2 \psi_j h_1^0 - \rho_k \sigma_1 \pi_2 \psi_j h\|_k$,

where $\|\cdot\|_k$ is the norm on $H_0^S(U_k)$.

Let $X_{ik} = \rho_k \pi_2 \psi_j h_1^0 - \rho_k \sigma_1 \psi_j h$. If $k = 1$, corollary 30
 tells us that there is a constant C_1 , such that $\|X_{ik}\|_k$
 $\leq C_1 \|\pi_2 \psi_j h_1^0 - \sigma_1 \psi_j h\|_k = C_1 \varepsilon$. For any k , $\rho_k \pi_2 \psi_j h_1^0$
 and $\rho_k \sigma_1 \psi_j h$ have their support included in: $U_k \cap U_1$,
 so they are elements of $H_0^S(U_1)$ and there is a constant
 C_{ik} such that $\|X_{ik}\|_k \leq C_{ik} \|X_{ik}\|_1$. Hence
 $\|h_1^\infty - \sigma_1 h\| = \sum_k \|X_{ik}\| \leq \sum_k C_1 C_{ik} \varepsilon$.

Therefore, $\|h^\infty - h\| \leq \sum_{i,k} C_1 C_{ik} \varepsilon$, and since ε
 can be made arbitrarily small, $C^\infty(E)$ is dense in $H^S(E)$.
 q.e.d.

Proof of B-Lemma: A norm on $H^S(E)$ is defined via an inner product on $J^S(E)$. We shall construct several inner products which are "locally flat" on various neighborhoods of M . Each will give a norm on $H^S(E)$ and, of course, all the norms are equivalent. Therefore, if a set P is bounded in one such norm it is bounded in all of them. We shall show that this implies that P is bounded in $H^{\circ S}(E)$ as well.

Let $\{V_i^k\}$ be a finite collection of convex coordinate neighborhoods on M such that:

1) $E|_{V_{ik}}$ is a trivial bundle for each V_{ik} .

2) For each K there exists a neighborhood U_K contained in a single neighborhood V_{i_0k} which we shall call V_{i_0k} , such that $U_K \cap V_{ik} = \emptyset$ if $i \neq i_0$ and $\bigcup_i V_{ik} = M$.

3) The sets U_K can be used to form a collection $\{(U_K, K_K, \rho_K), (V_j, \psi_j)\}$ which defines a norm for $H^S(E)$, where ψ_j is the restriction of the trivialization of E over V_{i_0k} .

Fix k , and let $\{\sigma_i\}$ be a partition of unity subordinate to $\{V_{ik}\}$, and let $\psi_1: E(V_{i_0k}) \rightarrow V_{i_0k} \times \mathbb{R}^m$ be a trivialization of $E(V_{i_0k})$. Let $\{e_1, \dots, e_m\}$ be a basis of \mathbb{R}^m and s_j , defined by $s_j(p) = \psi_1^{-1}(p, e_j)$ be

a section of $E(V_{ik})$. Then the set $\{S_j\}$ are a basis at each fibre of $E(V_{ik})$, and if $\{p_i\}$ are the elementary polynomials in the coordinates of V_{ik} , of degree less than or equal to s , the set of sections $\{p_i s_j\}$ is a basis of $J^s(E)(V_{ik})$ on each fibre. Let $\langle \rangle_i$ be the inner product of $J^s(E)$ over V_{ik} which makes $\{p_i s_j\}$ an orthonormal set on each fibre. Then $\sigma_i \langle \rangle_i$ is a smooth positive indefinite inner product on $J^s(E)$, where $\langle \rangle_i$ is defined to be zero on the complement of V_{ik} . Let $\langle \rangle_k = \sum \sigma_i \langle \rangle_i$. It is a positive definite inner product on $J^s(E)$ because for any $p \in M$, $\sigma_i(p)$ is positive for some i . Also on U_k , $\{p_i s_i\}$ is an orthonormal basis since $\sigma_i = 0$ on U_k if $i \neq i_0$ and $\sigma_{i_0} \upharpoonright U_k \equiv 1$.

Each set V_{ik} has a natural measure $(dx^1 \dots dx^n)_{ik}$ induced by the coordinate system. Let a measure on M be defined to be $\rho = \sum_i \sigma_i (dx^1 dx^2 \dots dx^n)_{ik}$, and let $\| \cdot \|_k$ be the norm defined on $H^s(E)$ by $\langle \rangle_k$ and ρ .

Consider the set $\{(U_k, K_k, \rho_k), (V_i, \psi_i)\}$ and the norm it defines on $\overset{\circ}{H}^s(E)$.

Let $S \in \sum f_i s_i$ where $\{s_i\}$ are the basis of sections for $E(V_{i_0 k})$. Then

$$(\|S\|_k^0)^2 = \sum_{|\alpha| \leq s} \int_{U_k} (D^\alpha(\rho f_i))^2 dx^1 \dots dx^n.$$

By corollary 30, there is a constant C_k such that

$$(\|S\|_k^0)^2 \leq C_k \sum_{\substack{i \\ |\alpha| \leq s}} \int_{U_k} (D^\alpha f_i)^2.$$

$$\text{Let } B_k(S) = \sum_{\substack{i \\ |\alpha| \leq s}} \int_{U_k} (D^\alpha f_i)^2.$$

We shall show that $B_k(S) \leq \|S\|_k^2$. Let \bar{s} be the section of $J^s(E)$ induced by S . Fix $p = (x_0^1, \dots, x_0^n)$. By Taylor's theorem,

$$f_i(x) - \sum_{|\alpha| \leq s} \frac{1}{\alpha!} (D_p^\alpha f_i)(x-p)^\alpha \in I_p^s.$$

Let p_α be the monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$.

Then $\bar{s}_p = \sum_{\substack{i \\ |\alpha| \leq s}} s_i \frac{1}{\alpha!} (D_p^\alpha f_i) p_\alpha$ for all p in $V_{k_1 0}$.

$$\|S\|_k^2 = \int_M \langle \bar{s}, \bar{s} \rangle \geq \int_{U_k} \langle \bar{s}, \bar{s} \rangle = \int_{U_k} \sum_{\substack{i \\ |\alpha| \leq s}} \left(\frac{1}{\alpha!} (D_p^\alpha f_i) \right)^2 dx^1 \dots dx^n$$

since $\{s_i p_\alpha\}$ are an orthonormal basis on U_k , and $\rho = dx^1 \dots dx^n$ on U_k .

Therefore $B_k(S) \leq \beta_k \|S\|_k^2$ where $\beta_k = \max_{|\alpha| \leq s} \{\alpha!\}$

so $(\|S\|_k^0)^2 \leq \beta_k C_k \|S\|_k^2$. $\sum_k \|S\|_k^0$ is the norm of S

in $H^s(E)$. We see that it is bounded by $\sum_k \sqrt{\beta_k C_k} \|S\|_k$.

The lemma follows.

Proposition 64: $H^S(E) = \dot{H}^S(E)$ as a topological vector space.

Proof: By the A-lemma, $C^\infty(E)$ is dense in each space, and by the B-lemma, the identity map is continuous for the two topologies. Therefore, it extends to a continuous map (also the identity) $\text{id}: H^S(E) \rightarrow \dot{H}^S(E)$ this map is injective, so to show isomorphism we need only show it is onto. Let U_k, V_{ik} be as in the B-lemma, and let $\{\sigma_k\}$ be a partition of unity subordinate to U_k , such that each σ_k is the square of a smooth function. Let $h \in \dot{H}^S(E)$. Then $\pi_2 \circ \psi_j \circ h \upharpoonright U_k$ is an H^S function on U_k , so $\sigma_k(\pi_2 \circ \psi_j \circ h) \upharpoonright U_k \in H^S_0(U_k)$ by lemma 37. Let $\{f_n\} \subseteq C^\infty_0(U_k)$ such that $f_n \rightarrow \sigma_k(\pi_2 \circ \psi_j \circ h) \upharpoonright U_k$ in $H^S_0(U_k)$. Let $\{h_n^k\} \subseteq C^\infty(E)$ by

$$h_n^k(p) = \begin{cases} \psi_j^{-1}(f_n(p), p) & p \in U_k \\ \psi_j^{-1}(0, p) & p \notin U_k \end{cases}$$

Then it is clear from the definition of $\|\cdot\|_k$ that since $f_n \rightarrow \sigma_k(\pi_2 \circ \psi_j \circ h) \upharpoonright U_k$, in $H^S_0(U_k)$, $h_n^k \rightarrow \sigma_k h$ in $\|\cdot\|_k$ on $H^S(E)$, so $\sigma_k h \in H^S(E)$. But $h = \sum_k \sigma_k h$ so $h \in H^S(E)$. Therefore, $\text{id}: H^S(E) \rightarrow \dot{H}^S(E)$ is onto and hence it is an isomorphism. q.e.d.

In order to define and discuss the properties of the Riemannian metric $\mu: \mathcal{M}^S \rightarrow B^0(H^S(S^2 T^*))$, we shall need to use differential operators. We now define such operators

and mention a few of their properties. Details can be found in 17, Chapter IV.

Definition 65: Let E and F be vector bundles over M . A map $D: C^\infty(E) \rightarrow C^\infty(F)$ is a differential operator of order k if D is a linear map such that if $s \in C^\infty(E)$ and $s \in Z_p^k(E)$, then $D(s)_p = 0$.

Jet bundles are a "universal space" for differential operators in the following sense: If $D: C^\infty(E) \rightarrow C^\infty(F)$ is a k^{th} order differential operator, then there is a linear map $\tilde{D}: J^k(E) \rightarrow F$ such that if $j_k: C^\infty(E) \rightarrow C^\infty(J^k(E))$ is the natural injection, then $D = \tilde{D} \cdot j_k$. Because of this, D can be uniquely extended to a continuous linear map $D: H^s(E) \rightarrow H^{s-k}(F)$ for all $s \geq k$. Also, given $\tilde{D}: J_k(E) \rightarrow F$ the map $D: C^\infty(E) \rightarrow C^\infty(F)$ defined by $D(s) = \tilde{D} \circ j_k(s)$ is a k^{th} order differential operator. We note that a differential operator of order k is also a differential operator of $k+1$ for any positive integer i , and also that the composition of two operators of order k and l is an operator of order $k+l$.

We give several examples of differential operators which we will have occasion to use:

Example 66: Let $\nabla: C^\infty(T(M)) \times C^\infty(E) \rightarrow C^\infty(E)$ be an affine connection on E . That is ∇ is a bilinear map

such that if $f \in C^\infty(M, \mathbb{R})$, $V \in C^\infty(T(M))$, $S \in C^\infty(E)$,

then $\nabla_{fV} S = f \nabla_V S$ and $\nabla_V (fS) = (Vf)S + f \nabla_V S$

$D: C^\infty(E) \rightarrow C^\infty(E \times T^*(M))$ by $D(X)(Y) = \nabla_Y X$ ($X \in C^\infty(E)$, $Y \in C^\infty(T(M))$).

D is a first order differential operator called the co-variant derivative (see 17, p. 85).

Example 67: Fix a Riemannian metric γ on M . Then

$\gamma \in C^\infty(S^2 T^*)$. Let $\alpha: C^\infty(T(M)) \rightarrow C^\infty(S^2 T^*)$ be defined by

$\alpha(X) = \theta_X(\gamma)$, the Lie derivative of γ with respect to X .

If in local coordinates $\gamma = g_{ij} dx^i \otimes dx^j$ and $X = v^i \frac{\partial}{\partial x^i}$,

then $\theta_X(\gamma) = (v^k \frac{\partial g_{ij}}{\partial x^k} + g_{kj} \frac{\partial v^k}{\partial x^i} + g_{ik} \frac{\partial v^k}{\partial x^j}) dx^i \otimes dx^j$

(see 9, p. 107).

If $X \in Z_p^1(T(M))$, $v_p^i = 0$ and $\frac{\partial v^k}{\partial x^i} \Big|_p = 0$ so $\alpha(X)_p = 0$.

Therefore, α is a first order differential operator.

Example 68: Fix γ as in example 67, and let SH be the

bundle of homomorphisms of $T(M)$ into $T(M)$, which are

symmetric with respect to γ ; that is, if $p \in M$,

$A \in SH_p$ and $X, Y \in T_p(M)$, $\gamma_p(AX, Y) = \gamma_p(X, AY)$. SH is

a subbundle of $\text{Hom}(T(M), T(M)) = T(M) \otimes T^*(M)$. We shall

construct a first order differential operator

$\beta: C^\infty(SH) \rightarrow C^\infty(S^2 T^* \otimes T(M))$.

* Whenever we use local coordinates we assume that repeated indices are summed.

We first recall that the exponential map on matrices (call it \exp), takes the set of symmetric matrices onto the set of positive definite symmetric matrices and is a diffeomorphism between these two sets (with their usual manifold structure). The exponential map can be applied to any linear transformation on finite dimensional vector spaces, so it induces a map $\exp: \text{Hom}(T(M), T(M)) \rightarrow \text{Hom}(T(M), T(M))$. $\exp(SH) \subseteq SH$, and $\exp \upharpoonright SH$ is a smooth injective map onto the positive definite elements of SH , where we say $A \in SH_p$ is positive definite if the inner product on $T_p(M)$ defined by $X, Y \rightarrow \gamma_p(AX, Y)$ is positive definite.

Let $s \in C^\infty(SH)$ and define $a \in C^\infty(SH)$ by $a_p = \exp(s_p)$. Then $\gamma * a$ defined by $(\gamma * a)_p(X, Y) = \gamma_p(a_p X, Y)$ is a Riemannian metric on M . Let ∇ and ∇^a be the Riemannian connections associated to γ and $\gamma * a$ respectively, (see 11, p. 71 for the definition of ∇). We define $\beta(s) \in C^\infty(S^2 T^* \otimes T(M))$ to be the difference tensor of ∇^a and ∇ ; that is, if $X, Y \in C^\infty(T(M))$, $\beta(s)(X, Y) = \nabla_X^a Y - \nabla_X Y$. $\beta(s)$ is symmetric in X and Y because ∇^a and ∇ are symmetric connections (see 11, p. 64).

We now show that β is a first order linear differential operator. To do so we use local coordinates. Let $\gamma \sim g_{ij}^*$, $s \sim s_{ij}^1$, $a \sim a_{ij}^1$ and Γ_{ij}^k and ${}^a\Gamma_{ij}^k$ be

* In general $x \sim x_{ij}$ means that the tensor field X is expressed locally by the functions $\{x_{ij}\}$.

the Riemannian-Christoffel symbols corresponding to ∇ and ∇^a respectively. Then $\beta(s) \sim {}^a\Gamma_{ij}^k - \Gamma_{ij}^k$.

From the Levi-Civita formula we know:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \text{ where } g^{kl} \text{ is the matrix}$$

inverse of g_{kl} , (see 11, p. 72). Substituting

$\gamma * a \sim a_i^r g_{rj}$ for γ we get

$$(+)\quad {}^a\Gamma_{ij}^k = \frac{1}{2} g^{ks} b_s^l \left(\frac{\partial a_l^r g_{rj}}{\partial x^i} + \frac{\partial a_l^r g_{ri}}{\partial x^j} - \frac{\partial a_i^r g_{rj}}{\partial x^l} \right)$$

where b_s^l is the matrix inverse of a_s^l . By direct computation we get:

$$\begin{aligned} {}^a\Gamma_{ij}^k - \Gamma_{ij}^k &= \frac{1}{2} g^{ks} b_s^l \left(\frac{\partial a_l^r}{\partial x^i} g_{rj} + \frac{\partial a_l^r}{\partial x^j} g_{ri} - \frac{\partial a_i^r}{\partial x^l} g_{rj} - a_i^r \frac{\partial g_{rj}}{\partial x^l} \right) \\ &\quad + \frac{1}{2} g^{kl} \frac{\partial g_{ij}}{\partial x^l}. \end{aligned}$$

$$\text{Note that } \frac{\partial a_i^j}{\partial x^k} = \frac{\partial (\exp(s))_i^j}{\partial x^k} = \exp(s)_i^l \frac{\partial s_l^j}{\partial x^k}.$$

$$\text{Therefore if } (s_i^j)_p = 0 \text{ and } \left. \frac{\partial s_i^j}{\partial x^k} \right|_p = 0,$$

$$\begin{aligned} ({}^a\Gamma_{ij}^k - \Gamma_{ij}^k)_p &= -\frac{1}{2} (g^{ks} b_s^l a_i^r \frac{\partial g_{rj}}{\partial x^l} - g^{kl} \frac{\partial g_{ij}}{\partial x^l})_p \\ &= -\frac{1}{2} (g^{ks} \delta_s^l \delta_i^r \frac{\partial g_{rj}}{\partial x^l} - g^{kl} \frac{\partial g_{ij}}{\partial x^l})_p \delta_i^j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \\ &= 0. \end{aligned}$$

Hence β is first order, and $\beta(s) = \tilde{\beta} \circ j_1(s)$ where $j_1: C^\infty(SH) \rightarrow C^\infty(J^1(SH))$ and $\tilde{\beta}: J^1(SH) \rightarrow S^2T^*$ is some bundle map.

To show that β is linear we have to show that $\tilde{\beta}$ is linear or that for any p in M , the map $\tilde{\beta}: J_p^1(SH) \rightarrow S^2T_p^*$ is linear. Fix some $p \in M$. s_p is a symmetric linear transformation so it is conjugate to a diagonal map --- a map $t_i^j = \lambda^i \delta_i^j$. Hence we can choose a coordinate system so that at p , $s_i^j = \lambda^i \delta_i^j$. With this coordinate system, $a_i^j = e^{\lambda^i} \delta_i^j$ and $b_i^j = e^{-\lambda^i} \delta_i^j$ at p . Hence at p ,

$$\begin{aligned} a \Gamma_{ij}^k - \Gamma_{ij}^k &= \frac{1}{2} g^{kl} \left(\frac{\partial s_l^r}{\partial x^i} g_{rj} + \frac{\partial s_l^r}{\partial x^j} g_{ri} \right) \\ &\quad - \frac{1}{2} g^{kl} \left(\delta_g^l \delta_i^t \frac{\partial s_t^r}{\partial x^l} g_{rj} + \delta_q^l \delta_i^r \frac{\partial g_{rj}}{\partial x^l} - \frac{\partial g_{ij}}{\partial x^q} \right) \end{aligned}$$

From this, it is clear that $\tilde{\beta} \upharpoonright J_p^1(SH)$ maps zero into zero and is homogeneous (i.e., $\tilde{\beta}(\lambda s) = \lambda \tilde{\beta}(s)$). However we cannot say immediately that $\tilde{\beta}$ is linear, for we do not know that given s and s' there is a single coordinate system for which they are both diagonal. But we do have the following:

Lemma 69: If a C^1 map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is homogeneous and $f(0) = 0$ then f is linear.

Proof: Fix $a \in \mathbb{R}^n$ and assume $m = 1$, or f is real valued.

$$f(a) = \frac{1}{\lambda}(f(\lambda a)) .$$

By the mean value theorem,

$$\frac{1}{\lambda}f(\lambda a) = \frac{1}{\lambda}(f(0) + D_{t\lambda_a} f(\lambda a)) \quad 0 \leq t \leq 1 .$$

Let $\lambda \rightarrow 0$ we get $f(a) = f(0) + \frac{1}{\lambda} D_0 f(\lambda a) = D_0 f(a)$.

Since $D_0 f$ is linear, so is f . If $m > 1$, we let $f = (f_1, f_2, \dots, f_m)$. Each f_i is linear, so f is linear.

q.e.d.

From the formula (+) and the fact that "exp" is smooth it is clear that $\tilde{\beta} \vdash J_p^1(\text{SH})$ is smooth. Hence by the above lemma, $\tilde{\beta}$ is linear so β is a first order linear differential operator.

$\mu: \mathcal{M} \rightarrow B(H^S(S^2 T^*))$ is defined by assigning to each $\gamma \in \mathcal{M}$ an inner product on $J^S(S^2 T^*)$. We proceed to show how this is done. First we note that for any bundle E and any positive k , there is an exact sequence of bundles $0 \rightarrow S^k T^* \otimes E \xrightarrow{i_k} J^k(E) \xrightarrow{j_k} J^{k-1}(E) \rightarrow 0$ where j_k is the natural projection $C^\infty(E)/Z_p^k(E) \rightarrow C^\infty(E)/Z_p^{k-1}(E)$ induced by the inclusion $Z_p^k(E) \subseteq Z_p^{k-1}(E)$ and $i_k: S^k T^* \otimes E \rightarrow J^k(E)$ is defined as follows:

Given $e \in E_p$ and $t \in T_p^*(M)$, pick $f \in C^\infty(M, \mathbb{R})$ and $s \in C^\infty(E)$ such that $s_p = e$ and $df_p = t$. Then

$i_k(t^k \otimes S_p)$ is the image of $f^k_s \in C^\infty(E)$ in $J^k_p(E)$.

For the proof that i_k is well defined (i.e., independent of the choice of f and s) and that the sequence is exact, see 17, p. 58.

Proposition 70: $D: C^\infty(E) \rightarrow C^\infty(T^*(M) \otimes E)$ is a covariant derivative, then $\tilde{D}: J^1(E) \rightarrow T^*(M) \otimes E$ is a splitting of the exact sequence $T^*(M) \otimes E \rightarrow J^1(E) \rightarrow J^0 E = E$ and conversely any such splitting defines a covariant derivative.

Proof: See 17, pp. 84-85.

Proposition 71: If D_1 and D_2 are covariant derivatives for bundles E and F , then

$D: C^\infty(E \otimes F) \rightarrow C^\infty(T^*(M) \otimes E \otimes F)$ defined by

$D(e \otimes f) = D_1 e \otimes f + \pi_{12}(e \otimes D_2 f)$ (where

$\pi_{12}: E \otimes T^*(M) \otimes F \rightarrow T^*(M) \otimes E \otimes F$ by interchanging $T^*(M)$ and E factors) is a covariant derivative for $E \otimes F$.

Proof: See 17, pp. 87-88.

Remark: If $E = F$ and $D_1 = D_2$, then

$D \upharpoonright C^\infty(S^2 E): C^\infty(S^2 E) \rightarrow C^\infty(T^* \otimes S^2 E)$ and $D \upharpoonright C^\infty(S^2 E)$

is a covariant derivative.

Corollary 72: Given a finite set of bundles $\{E_i\}_{i=1}^k$, each with covariant derivative D_i , there is a covariant derivative D on $E_1 \otimes E_2 \otimes \cdots \otimes E_k$ such that

$D(e_1 \otimes e_2 \otimes \cdots \otimes e_n) = \sum_{i=1}^k \pi_{1i}(e_1 \otimes e_2 \otimes \cdots \otimes D_i e_1 \otimes \cdots \otimes e_k)$
 where π_{1i} interchanges the first and i^{th} factors.

Note: The above D will be written as $D_1 \otimes D_2 \otimes \cdots \otimes D_k$.

Now given covariant derivatives D on T^* and D_0 on $S^2 T^*$, we get a covariant derivative (call it D_k) on $(T^*)^k \otimes S^2 T^*$. Let Δ_k be the k^{th} order differential operator $\Delta_k: C^\infty(S^2 T^*) \rightarrow C^\infty((T^*)^k \otimes S^2 T^*)$ defined by $\Delta_k = D_k \circ D_{k-1} \circ \cdots \circ D_0$. Δ_k induces a linear bundle map $\tilde{\Delta}_k: J^k(S^2 T^*) \rightarrow (T^*)^k \otimes S^2 T^*$. Let $S_k: (T^*)^k \rightarrow S^k T^*$ be the canonical map which "symmetrizes" tensors:

$$S_k(t_1 \otimes \cdots \otimes t_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} t_{\sigma(1)} \otimes \cdots \otimes t_{\sigma(k)} \quad \text{where } \mathfrak{S}_k$$

is the symmetric group on k symbols.

Let $\tilde{S}\tilde{\Delta}_k: J^k(S^2 T^*) \rightarrow S^k T^* \otimes S^2 T^*$ be defined by:

$$\tilde{S}\tilde{\Delta}_k = (S_k \otimes \text{Id}_{S^2 T^*}) \circ \tilde{\Delta}_k$$

Proposition 73: $\tilde{S}\tilde{\Delta}_k: J^k(S^2 T^*) \rightarrow S^k T^* \otimes S^2 T^*$ splits the exact sequence $S^k T^* \otimes S^2 T^* \rightarrow J^k(S^2 T^*) \rightarrow J^{k-1}(S^2 T^*)$.

Proof: See 17, p. 90.

Corollary 74: Covariant derivatives on $T^*(M)$ and $S^2 T^*$ induce an isomorphism:

$$J^k(S^2 T^*) \cong \sum_{i=0}^k S^i T^* \otimes S^2 T^*.$$

Proof: The splitting of the above proposition gives an isomorphism $J^1(S^2T^*) \cong S^1T^* \otimes S^2T^* \oplus J^{1-1}(S^2T^*)$. The corollary follows from an induction on i .

$\gamma \in \mathcal{M}$ is an inner product on $T(M)$; using it we shall construct an inner product on $J^k(S^2T^*)$.

γ defines an isomorphism (also called γ).

$\gamma: T(M) \rightarrow T^*(M)$ by: $\gamma(V)$ is the map $X \rightarrow (\gamma(V), X)$.

By means of this isomorphism it gives an inner product on $T^*(M)$. Also γ gives an inner product on $(T^*)^k$ defined by: $(V_1 \otimes V_2 \otimes \cdots \otimes V_k, W_1 \otimes W_2 \otimes \cdots \otimes W_k) = \prod_{i=1}^k (V_i, W_i)$

and therefore on the subbundle S^kT^* . Hence γ gives an inner product on $S^1T^* \otimes S^2T^*$ and on $\sum_{i=0}^k S^iT^* \otimes S^2T^*$.

Therefore, we seek an isomorphism between $J^k(S^2T^*)$ and $\sum_{i=0}^k S^iT^* \otimes S^2T^*$. By corollary 74, it is sufficient to

find covariant derivatives on S^2T^* and $T^*(M)$.

Given $\gamma \in \mathcal{M}$, we have a canonical affine connection ∇ on $T(M)$, the Riemannian connection. Also we have an isomorphism between $T(M)$ and $T^*(M)$ so we get a connection ∇' on $T^*(M)$.

∇' is defined by $\nabla'_V \omega = \gamma(\nabla_V(\gamma^{-1}\omega))$ where $V \in C^\infty(T(M))$, $\omega \in C^\infty(T^*(M))$. ∇' induces a covariant derivative on $T^*(M)$ and by corollary 72, we get one on S^2T^* as well. Therefore γ defines an isomorphism

$\sum_{i=0}^s S^1 T^* \otimes S^2 T^* \cong J^s(S^2 T^*)$ and an inner product on $J^s(S^2 T^*)$.

γ also defines a measure on M --- in local coordinates it is $\sqrt{\det(\gamma_{ij})} dx^1 dx^2 \cdots dx^n$. Therefore γ induces an inner product on $H^0(J^s(S^2 T^*))$, and this gives an inner product on $H^s(S^2 T^*)$.

Definition 75: Let $\mu: \mathcal{M} \rightarrow B(H^s(S^2 T^*))$ by $\mu(\gamma)$ equals the inner product on $H^s(S^2 T^*)$ induced by γ in the manner described above.

We proceed to show that μ extends to \mathcal{M}^s , and that the extension is smooth. We begin by using the differential operator β to construct a k^{th} -order differential operator from $C^\infty(SH) \times C^\infty(S^2 T^*)$ to $C^\infty(S^k T^* \otimes S^2 T^*)$.

Fix $\gamma_0 \in \mathcal{M}$ and let SH be the subbundle of $\text{Hom}(T(M), T(M))$, consisting of homomorphisms which are symmetric with respect to γ_0 . Let $S\tilde{\Delta}^k: J^k(S^2 T^*) \rightarrow S^k T^* \otimes S^2 T^*$ be the homomorphism induced by γ_0 . Pick $a \in C^\infty(SH)$. Then $\gamma_0^*(\exp a) \in \mathcal{M}$. Let $S\tilde{\Delta}_a^k: J^k(S^2 T^*) \rightarrow S^k T^* \otimes S^2 T^*$ be the homomorphism induced by $\gamma_0^*(\exp a)$. Let $S\Delta^k, S\Delta_a^k: C^\infty(S^2 T^*) \rightarrow C^\infty(S^k T^* \otimes S^2 T^*)$ be the differential operators defined by these homomorphisms.

Proposition 76: Let $\beta_k: C^\infty(SH \otimes S^2 T^*) \rightarrow C^\infty(S^k T^* \otimes S^2 T^*)$ by $\beta_k(a \otimes g) = S\Delta^k(g) - S\Delta_a^k(g)$. β_k is a k^{th} order

differential operator.

Proof: We proceed by induction on k :

Case I: $k = 1$. We first analyse the connections on $T(M)$ and $T^*(M)$ induced by the metrics γ_0 and $\gamma = \gamma_0^*(\exp a)$. Let ∇ and ∇^a be the connections on $T(M)$; $'\nabla$ and $'\nabla^a$ the connections on $T^*(M)$ and $'D$, $'D_a$ their covariant derivatives. We recall that for any Riemannian connection ∇ , $X(V, W) = (\nabla_X V, W) + (V, \nabla_X W)$.

Lemma 77: Let $\omega \in C^\infty(T^*(M))$. Then $'D\omega - 'D_a\omega = \beta(a)(\omega)$. Note that $\beta(a) \in C^\infty(S^2T^* \otimes T)$, so $\beta(a)(\omega) \in C^\infty(S^2T^* \subseteq C^\infty((T^*)^2))$.

Proof of lemma: Pick $X, V \in C^\infty(T(M))$.

$$\begin{aligned} 'D\omega(X, V) - 'D_a\omega(X, V) &= \gamma_0(\nabla_X(\gamma_0(\omega)), V) - \gamma(\nabla_X^a(\gamma(\omega)), V) \\ &= X(\omega(V)) - \omega(\nabla_X V) - X(\omega(V)) + \omega(\nabla_X^a V) \\ &= \omega(\beta(a)(X, V)) . \end{aligned}$$

The lemma follows.

Now let D and D^a be covariant derivatives on S^2T^* induced by γ_0 and γ . Using corollary 72,

$$\begin{aligned} D(V \otimes W) - D^a(V \otimes W) &= ('D - 'D_a)(V) \otimes W + \pi_{12}(V \otimes ('D - 'D_a)(W)) \\ &= \beta(a)(V) \otimes W + \pi_{12}(V \otimes \beta(a)(W)) . \end{aligned}$$

$$\begin{aligned} \text{Therefore } \beta_1(a \otimes V \otimes W) &= (S\Delta^1 - S\Delta_a^1)(V \otimes W) \\ &= \beta(a)(V) \otimes W + \pi_{12}(V \otimes \beta(a)(W)) . \end{aligned}$$

Since β is a first order differential operator it is clear from the above formula that

$\beta_1: C^\infty(SH \otimes S^2 T^*) \rightarrow C^\infty(T^* \otimes S^2 T^*)$ is also.

Case II: We shall assume the proposition for β_k and prove it for β_{k+1} . Let D_1 and D_1^a be the covariant derivatives on $(T^*)^1 \otimes S^2 T^*$ induced by γ_0 and γ . Then $\tilde{S}\Delta^k, \tilde{S}\Delta_a^k: J^k(S^2 T^*) \rightarrow S^k T^* \otimes S^2 T^*$ are the linear maps corresponding to the k^{th} order operators

$$S_k \circ \Delta^k = S_k \circ D_k \circ D_{k-1} \circ \cdots \circ D_1 \quad \text{and}$$

$$S_k \circ \Delta_a^k = S_k \circ D_k^a \circ D_{k-1}^a \circ \cdots \circ D_1^a.$$

$$\begin{aligned} S\Delta^{k+1} - S\Delta_a^{k+1} &= S_{k+1} \circ (D_{k+1} - D_{k+1}^a) \circ \Delta^k + S_{k+1} \circ D_{k+1}^a \\ &\quad \circ (\Delta^k - \Delta_a^k). \end{aligned}$$

We look at the two terms on the left of the above expression.

In the first term,

$$\begin{aligned} (D_{k+1} - D_{k+1}^a)(v_1 \otimes v_2 \otimes \cdots \otimes v_{k+1} \otimes v_{k+2}) = \\ \sum_{i=1}^{k+2} \pi_{1i}(v_1 \otimes \cdots \otimes \beta(a)(v_i) \otimes \cdots \otimes v_{k+1} \otimes v_{k+2}) \end{aligned}$$

where $v_1 \otimes \cdots \otimes v_k \in C^\infty((T^*)^k)$, $v_{k+1} \otimes v_{k+2} \in C^\infty(S^2 T^*)$

and π_{1i} interchanges the first and i^{th} factors. As

in case I, $a \otimes v_1 \otimes \cdots \otimes v_{k+2} \rightarrow (D_{k+1} - D_{k+1}^a)(v_1 \otimes \cdots \otimes v_{k+2})$

is a first order operator from

$C^\infty(SH \otimes (T^*)^k \otimes S^2 T^*) \rightarrow C^\infty((T^*)^{k+1} \otimes S^2 T^*)$, and it is

induced by an element of $\text{Hom}(J^1(\text{SH} \otimes (T^*)^k \otimes S^2 T^*), (T^*)^{k+1} \otimes S^2 T^*)$.
 Also Δ^k gives an element of $\text{Hom}(J^k(S^2 T^*), (T^*)^k \otimes S^2 T^*)$.
 Hence $S_{k+1} \circ (D_{k+1} - D_{k+1}^a) \circ \Delta_k$ is defined by an element
 of $\text{Hom}(J^1(\text{SH}) \otimes J^k(S^2 T^*), S^{k+1} T^* \otimes S^2 T^*)$ so it is $(k+1)$
 order operator from $C^\infty(\text{SH} \otimes S^2 T^*)$ to $C^\infty(S^{k+1} T^* \otimes S^2 T^*)$.

Second term: By assumption, $S_k \circ (\Delta^k - \Delta_a^k)$ is a k^{th}
 order linear operator on $C^\infty(\text{SH} \otimes S^2 T^*)$. Hence $\Delta^k - \Delta_a^k$
 induces an element of $\text{Hom}(J^k(\text{SH} \otimes S^2 T^*), (T^*)^k \otimes S^2 T^*)$
 which gives an element of $\text{Hom}(J^{k+1}(\text{SH} \otimes S^2 T^*), J^1((T^*)^k \otimes S^2 T^*))$.
 Also $S_{k+1} \circ \tilde{D}_{k+1}^a: J^1((T^*)^k \otimes S^2 T^*) \rightarrow S^{k+1} T^* \otimes S^2 T^*$
 is a linear map.

Hence $S_{k+1} \circ D_{k+1}^a \circ (\Delta^k - \Delta_a^k)$ is induced by an
 element $\tilde{\beta}_{k+1}$ of $\text{Hom}(J^{k+1}(\text{SH} \otimes S^2 T^*), S^{k+1} T^* \otimes S^2 T^*)$
 so it is a $(k+1)$ order operator from $C^\infty(\text{SH} \otimes S^2 T^*)$ to
 $C^\infty(S^{k+1} T^* \otimes S^2 T^*)$, which is $(a \otimes q) \rightarrow S_{k+1} \circ D_{k+1}^a$
 $\circ (\Delta^k - \Delta_a^k)(q)$.

Hence each term of β_{k+1} is a differential operator
 of order $k+1$. This concludes the proof.

Since β_s is an s^{th} order differential operator,
 $\beta_s: H^s(\text{SH} \otimes S^2 T^*) \rightarrow H^0(S^s T^* \otimes S^2 T^*)$ is a continuous linear
 map.

For any $k \leq s$, $\beta_k: C^\infty(\text{SH} \otimes S^2 T^*) \rightarrow C^\infty(S^k T^* \otimes S^2 T^*)$
 is a differential operator of order s . Therefore, the

same is true of $\mathfrak{b}: C^\infty(SH \otimes S^2 T^*) \rightarrow C^\infty(\sum_{k=0}^S S^k T^* \otimes S^2 T^*)$

where \mathfrak{b} is defined to be $\bigoplus_{k=0}^S \beta_k$. Hence

$$\mathfrak{b}: H^S(SH \otimes S^2 T^*) \rightarrow H^0(\sum_{k=0}^S S^k T^* \otimes S^2 T^*) .$$

We note that the map $\mu: \mathcal{M} \rightarrow B(H^S(S^2 T^*))$ depends on three things:

- 1) The isomorphism $J^S(S^2 T^*) \hookrightarrow \sum_{k=0}^S S^k T^* \otimes S^2 T^*$.
- 2) The inner product on $\sum_{k=0}^S S^k T^* \otimes S^2 T^*$.
- 3) The measure on M .

Therefore, to prove that μ is smooth we look at each of these separately.

1) By fixing $\gamma_0 \in \mathcal{M}$, we got a correspondence $E: \mathcal{M}^S \longleftrightarrow H^S(SH)$ defined by $\gamma = \gamma_0^* a \rightarrow \log a$.

Since \exp is a diffeomorphism from symmetric to positive definite symmetric matrices, example 48 tells us that this correspondence is a diffeomorphism. Therefore to show μ is smooth we can work either on \mathcal{M}^S or on $H^S(SH)$.

The standard map $q: SH \times S^2 T^* \rightarrow SH \otimes S^2 T^*$ is bilinear.

Hence by lemma 29, there is an induced map

$\tilde{q}: H^S(SH) \times H^S(S^2 T^*) \rightarrow H^S(SH \otimes S^2 T^*)$ which is continuous and bilinear. $\mathfrak{b} \circ \tilde{q}: H^S(SH) \times H^S(S^2 T^*) \rightarrow H^0(\sum_{k=0}^S S^k T^* \otimes S^2 T^*)$

is therefore also continuous and bilinear.

Let $\tilde{\mu}_1: H^S(SH) \rightarrow L(H^S(S^2T^*), H^0(\sum_{k=0}^S S^kT^* \otimes S^2T^*))$
be defined by $\tilde{\mu}_1(a)(g) = \mathfrak{B} \circ \tilde{q}(a, q)$, then $\tilde{\mu}_1$ is clearly
a continuous linear map. Also from the construction of \mathfrak{B}
we know the following:

Proposition 78: Let $a \in C^\infty(SH)$, $\gamma = \gamma_0^*(\exp a)$, and let
 $h_0, h: C^\infty(S^2T^*) \rightarrow C^\infty(\sum_{k=0}^S S^kT^* \otimes S^2T^*)$ be the linear maps
induced by γ_0 and γ respectively. Then for all
 $g \in C^\infty(S^2T^*)$, $\tilde{\mu}_1(a)(g) = h_0(g) - h(g)$.

Proof: Simply follow the steps of the construction of \mathfrak{B} .

Definition: 79: $\mu_1: \mathcal{M}^S \rightarrow L(H^S(S^2T^*), H^0(\sum_{k=0}^S S^1T^* \otimes S^2T^*))$
by $\mu_1(\gamma) = h_0 - \tilde{\mu}_1(E^{-1}(\gamma))$. μ_1 is smooth since E^{-1} and
 $\tilde{\mu}_1$ are. Also $\mu_1(\gamma)(g) = h(g)$ by the above proposition.

2) We know that since every $\gamma \in \mathcal{M}^S$ is continuous,
it defines a continuous inner product on $T^*(M)$ and hence
on $\sum_{k=0}^S S^kT^* \otimes S^2T^*$. Let PR be the space of continuous
inner products on $\sum_{k=0}^S S^kT^* \otimes S^2T^*$ and let $\mu_2: \mathcal{M}^S \rightarrow PR$
be defined by: $\mu_2(\gamma)$ is the inner product on
 $\sum_{k=0}^S S^kT^* \otimes S^2T^*$ induced by γ . Note that PR is an open
convex cone in the linear space of continuous bilinear maps
on $\sum_{k=0}^S S^kT^* \otimes S^2T^*$.

Proposition 80: $\mu_2: \mathcal{M}^S \rightarrow PR$ is smooth.

Proof: We follow several easy steps:

a) Let \mathcal{N}^S be the space of H^S Riemannian structures on $T^*(M)$ (\mathcal{N}^S is an open convex cone in $H^S(S^2T)$). Let $i: \mathcal{M}^S \rightarrow \mathcal{N}^S$ by $i(\gamma)$ is the inner product on $T^*(M)$ defined by γ . In local coordinates $i(g_{ij} dx^i \otimes dx^j) = g^{ij} X_i X_j$ where g^{ij} is the matrix inverse of g_{ij} . Since matrix inverse is a smooth map, by example 48, we know that $i(\gamma)$ is H^S when γ is and that the map i is smooth.

b) For any bundle E , let $\mathcal{N}^S(E)$ be the space of H^S inner products on E . Let $j_2: \mathcal{N}^S(T^*) \rightarrow \mathcal{N}^S(T^* \otimes T^*)$ be defined by: $j_2(g)(V \otimes W, V' \otimes W') = g(V, V')g(W, W')$. j_2 is the composition of the diagonal map $\Delta: \mathcal{N}^S \rightarrow \mathcal{N}^S \times \mathcal{N}^S$ and a bilinear map. It is smooth by lemma 29.

Similarly $j_k: \mathcal{N}^S \rightarrow \mathcal{N}^S((T^*)^k)$ is smooth.

c) The map $q_k: \mathcal{N}^S(T^{*k}) \rightarrow \mathcal{N}^S(S^{kT^*})$ defined by restricting an inner product on $(T^*)^k$ to a product on the subbundle S^{kT^*} gives a continuous linear map $\tilde{q}_k: H^S(S^2(T^{*k})^*) \rightarrow H^S(S^2(S^{kT^*})^*)$.

d) Combining j_k and \tilde{q}_k , we define $j: \mathcal{N}^S(T^*) \rightarrow \mathcal{N}^S(\sum_{k=0}^S S^{kT^*} \otimes S^{2T^*})$, $j(g)$ being the inner product on $\sum_{k=0}^S S^{kT^*} \otimes S^{2T^*}$ induced by the inner product g on $T^*(M)$. j is smooth because \tilde{q}_k and j_k are.

e) Let $r: \mathcal{M}^S(\sum_{k=0}^S S^{kT*} \otimes S^{2T*}) \rightarrow PR$ be the inclusion of H^S maps into C^0 maps. r is a continuous inclusion $H^S(S^2(\sum_{k=0}^S S^{kT*} \otimes S^{2T*})^*) \subseteq C^0(S^2(\sum_{k=0}^S S^{kT*} \otimes S^{2T*})^*)$ so it is smooth.

f) $\mu_2 = r \circ j \circ i$ so μ_2 is smooth. q.e.d.

3) We fix a smooth measure ρ_0 on M , ρ_0 being determined by the Riemannian metric γ_0 . Then the correspondence $\rho \sim f$, $f \in C^0(M, \mathbb{R})$, which is defined by $\rho = f\rho_0$, associates with each continuous measure a continuous function on M . Let $\mu_3: \mathcal{M}^S \rightarrow C^0(M, \mathbb{R})$ be defined locally $\mu_3(\gamma) = \sqrt{\det(\gamma_{ij})}/\sqrt{\det(\gamma_0)_{ij}}$ so that if ρ is the measure associated to γ , $\rho = \mu_3(\gamma)\rho_0$.

Proposition 81: μ_3 is smooth.

Proof: Let $\bar{\mu}_3: SH \rightarrow \mathbb{R} \times M$ by

$\bar{\mu}_3(a) = \log(\sqrt{\det(\gamma_0^* \exp a)}/\det(\gamma_0))$. This is a linear map from SH to $\mathbb{R} \times M$, the trivial bundle over M with fibre \mathbb{R} . Therefore $\bar{\mu}_3$ induces a continuous linear map $\bar{\mu}_3: H^S(SH) \rightarrow H^S(M, \mathbb{R})$. Let $e: H^S(M, \mathbb{R}) \rightarrow H^S(M, \mathbb{R})$ by $e(f) = e^f$ (i.e., $e(f)(p) = \sum_{i=0}^{\infty} f(p)^i/i!$). e is smooth. Let $i: H^S(M, \mathbb{R}) \hookrightarrow C^0(M, \mathbb{R})$ be the inclusion. Then $\mu_3 = i \circ e \circ \bar{\mu}_3 \circ E$ where $E: \mathcal{M}^S \rightarrow H^S(SH)$ is

the diffeomorphism defined in 1). Hence μ_3 is smooth.

Proposition 82: $\mu: \mathcal{M} \rightarrow B^0(H^S(S^{2T*}))$ extends to a map on \mathcal{M}^S and the extension is a smooth map.

Proof: Let $\bar{\mu}: L(J^S(S^{2T*}), \sum_{k=0}^S S^{kT*} \otimes S^{2T*}) \times PR \times C^0(M, \mathbb{R}) \rightarrow B(H^S(S^{2T*}))$ be the obvious map; that is, $\bar{\mu}(h, g, f)$ is the bilinear form induced on $H^S(S^{2T*})$ from the bilinear form on $H^0(J^S(S^{2T*}))$ which is defined by the measure $f\rho_0$, the inner product on $J^S(S^{2T*})$ which comes from the homomorphism h and the inner product g on $\sum_{k=0}^S S^{kT*} \otimes S^{2T*}$.

$\bar{\mu}$ is clearly a continuous tri-linear map.

$\bar{\mu} \circ (\mu_1, \mu_2, \mu_3): \mathcal{M}^S \rightarrow B^0(H^S(S^{2T*}))$ is therefore a smooth map. It has been constructed so that $\bar{\mu} \circ (\mu_1 \times \mu_2 \times \mu_3) \upharpoonright \mathcal{M} = \mu$.

The proposition follows.

To insure that $\mu: \mathcal{M}^S \rightarrow B^0(H^S(S^{2T*}))$ defines a Riemannian metric for \mathcal{M}^S we need only check that the image of μ consists solely of positive definite inner products which induce the given topology on $H^S(S^{2T*})$.

We note the following:

Lemma 83: Let E be a vector bundle. Then a norm on $H^0(E)$ is defined by a continuous inner product on E and a positive continuous measure on M . The topology on $H^0(E)$ is independent of the particular inner product

and measure which are chosen.

Proof: See 17, p. 148 (Palais states this lemma for smooth inner product and measure, but in his proof uses only the fact that they are continuous.)

It is clear that the images of μ_2 and μ_3 consist of continuous positive definite inner products and continuous positive measures, so together they induce the usual topology on $H^0(\sum_{k=0}^S S^k T^* \otimes S^{2-k} T^*)$. Hence to show the μ is a Riemannian metric, it is enough to show:

Proposition 84: $\mu: \mathcal{M}^S \rightarrow L(H^S(S^{2-k} T^*), H^0(\sum_{k=0}^S S^k T^* \otimes S^{2-k} T^*))$

has image consisting of maps which are homeomorphisms into.

Proof: $\mu_1(\gamma)$ is injective for all γ in \mathcal{M}^S for, if for some g in $H^S(S^{2-k} T^*)$, $\mu(\gamma)(g) = 0$ then the component of $\mu_1(\gamma)(g)$ which lies in $S^0 T^* \otimes S^{2-k} T^* = S^{2-k} T^*$ is zero. But this component is just g .

Since $\mu_1(\gamma)$ is injective, we need only show that for $\{g_n\}$ a sequence in $H^S(S^{2-k} T^*)$, $\mu_1(\gamma)(g_n) \rightarrow 0$ implies $g_n \rightarrow 0$. To show this we will use coordinates and examine the local situation. We note that if $\gamma = \gamma_0 * (\exp a)$, $\mu_1(\gamma)(g_n) = \sum_{k=0}^S S \Delta_a^k(g_n)$. For the moment, we assume the following statements:

1) Let $g \in H^s(S^{2T*})$, $k \leq s$. In local coordinates

$$\Delta_a^k(g) = \sum_{|\alpha|=k} D^\alpha(g_{ij}) + P_k + Q_k \quad \text{where } P_k \text{ is a sum of terms of form } f D^\alpha g_{ij}, \quad f \text{ is a } \begin{cases} C^{[s/2]+1-k/2} & k \text{ even} \\ C^{[s/2]-\frac{k-1}{2}} & k \text{ odd} \end{cases}$$

functions and $|\alpha| < k$; and Q_k is a sum of terms

$f' D^\alpha g_{ij}$, f' an H^{s-k} function and

$$|\alpha| \leq \begin{cases} k/2 & k \text{ even} \\ \frac{k+1}{2} & k \text{ odd} \end{cases}$$

2) If $H^{[s/4]} \subseteq C^0$, then

$g_n \rightarrow 0$ in $H^{k-1}(S^{2T*})$ and $S_k \Delta_a g_n \rightarrow 0$ in $H^0(S^{kT*} \otimes S^{2T*})$ implies $g_n \rightarrow 0$ in $H^k(S^{2T*})$.

We use 2) to prove the proposition. Note that

$\mu_1(\gamma)(g_n) \rightarrow 0$ means that for all $k \leq s$, $S \Delta_a^k(g_n) \rightarrow 0$ in $H^0(S^{kT*} \otimes S^{2T*})$.

$S_0 \Delta_a(g_n) = g_n$ so we know that $g_n \rightarrow 0$ in $H^0(S^{2T*})$.

Now the proposition follows by induction using 2).

Proof of 1): We use induction on k . $\Delta_a^0(g) = g$ so the case $k = 0$ is obvious. Note that we have chosen s large enough so that $H^s(S^{2T*}) \subseteq C^{[s/2]+1}(S^{2T*})$, and that $\Delta_a^{k+1}g = D_a^{k+1} \circ \Delta_a^k(g)$. We assume that

$$\Delta_a^k(g) = \sum_{|\alpha|=k} D^\alpha(g_{ij}) + P_k + Q_k. \quad \text{For any function } h$$

which is a coordinate function of a local section of $S^k_T \otimes S^{2k}_T$, $D_a^{k+1} h$ has coordinates of form $\frac{\partial h}{\partial x^1} + fh$ where f is an H^{s-1} (and therefore a $C^{[s/2]}$) function.

Therefore $D_a^{k+1} \left(\sum_{|\alpha|=k} D^\alpha(g_{ij}) \right) = \sum_{|\alpha|=k+1} D^\alpha g_{ij} + P_{k+1}$.

If $f D^\alpha(g_{ij}) \in P_k$, $D_a^{k+1} f D^\alpha g_{ij}$ has form

$$\frac{\partial f}{\partial x^\ell} D^\alpha g_{ij} + f \frac{\partial D^\alpha g_{ij}}{\partial x^\ell} + P_k \cdot \frac{\partial f}{\partial x^\ell} D^\alpha g_{ij} \text{ and } f \frac{\partial D^\alpha g_{ij}}{\partial x^\ell} \text{ are}$$

of form P_{k+1} . Similarly if $f' D^\alpha g_{ij}$ is of form Q_k ,

$D_a^{k+1}(f' D^\alpha g_{ij})$ is of form Q_{k+1} .

Proof of 2): By 1) $S_k \Delta_a^k(g_n) = \sum_{|\alpha|=k} D^\alpha g_{ij} + P_k(g_n) + Q_k(g_n)$.

But since $g_n \rightarrow 0$ in H^{k-1} , $P_k(g_n) \rightarrow 0$. We examine

a term $f' D^\alpha(g_{ij})_n$ of $Q_k(g_n)$. If $k \leq [s/2] + 1$,

$H^{s-k} \subseteq C^0$ so f' is C^0 , hence $(g_{ij})_n \rightarrow 0$ in H^{k-1}

and $|\alpha| \leq [k/2] + 1$ implies $Q_k(g_n) \rightarrow 0$. If

$k > [s/2] + 1$, the fact that $D^\alpha(g_{ij})_n \rightarrow 0$ in

$k-1-([k/2] + 1)$ implies $D^\alpha(g_{ij})_n \rightarrow 0$ in $H^{[s/4]}$ and

hence in C^0 . Therefore, $Q_k(g_n) \rightarrow 0$ in this case also.

$$P_k(g_n) + Q_k(g_n) \rightarrow 0 \text{ implies } \sum_{|\alpha|=k} D^\alpha(g_{ij})_n \rightarrow 0.$$

But this and the fact that $g_n \rightarrow 0$ in H^{k-1} implies

that $g_n \rightarrow 0$ in H^k . q.e.d.

V. The Group Action.

Now that we have constructed the group manifold \mathcal{D}^s and the Riemannian manifold of metrics, \mathcal{M}^s , we proceed to discuss the action of the group on the manifold of metrics.

Definition 85: Let $A: \mathcal{D}^{s+1} \times \mathcal{M}^s \rightarrow \mathcal{M}^s$ be defined by $A(\eta, \gamma) = \eta^*(\gamma)$, where $\eta^*(\gamma)$ is the metric such that at any $p \in M$, $X, Y \in T_p(M)$, $\eta^*(\gamma)_p(X, Y) = \gamma_{\eta(p)}(T_p\eta(X), T_p\eta(Y))$. Clearly A is a right action.

Lemma 86: The above definition makes sense; that is, $\eta \in \mathcal{D}^{s+1}$ and $\gamma \in \mathcal{M}^s$ implies $\eta^*(\gamma) \in \mathcal{M}^s$.

Proof: We use local coordinates: write η as $y^1(x^1 \dots x^n), y^2(x^1 \dots x^n), \dots y^n(x^1 \dots x^n)$ and γ as $\gamma_{ij} dy^i \otimes dy^j$.

$$\text{Then } \eta^*(\gamma)_p = (\gamma_{kl})_{\eta(p)} \left. \frac{\partial y^k}{\partial x^i} \right|_p \left. \frac{\partial y^l}{\partial x^j} \right|_p dx^i \otimes dx^j.$$

The set $\{y^i\}$ consists of H^{s+1} functions and $\{\gamma_{ij}\}$ are H^s functions. Therefore $\{\frac{\partial y^i}{\partial x^j}\}$ is a set of H^s functions. Since η is C^1 and has C^1 inverse $\gamma_{kl} \circ \eta$ is H^s by remark 1) following the composition lemma. Hence by corollary 30, $\gamma_{kl} \frac{\partial y^l}{\partial x^i} \frac{\partial y^k}{\partial x^j}$ is H^s as well. The lemma follows.

We can extend A to a map

$\tilde{A}: \mathcal{D}^{s+1} \times H^s(S^2 T^*) \rightarrow H^s(S^2 T^*)$ by defining

$\tilde{A}(\eta, g) = \eta^*(g)$ as above. In this case if we fix

$\eta \in \mathcal{D}^{s+1}$ and consider $\eta^*: H^s(S^2 T^*) \rightarrow H^s(S^2 T^*)$, we get the following:

Proposition 87: $\eta^*: H^s(S^2 T^*) \rightarrow H^s(S^2 T^*)$ is a continuous linear map.

Proof: Linearity is obvious. Let $\|\cdot\|_1, \|\cdot\|_3, \|\cdot\|_1, \|\cdot\|_3$ be norms for $H^s(S^2 T^*)$ corresponding to sets $U_1, W_1, \eta^{-1}(U_1)$ and $\eta^{-1}(W_1)$ as in Lemma 46. Looking at η^* in norms $\|\cdot\|_1$ and $\|\cdot\|_3$, it is locally the sequence.

$$H_0^s(U_1) \xrightarrow{(1)} H_0^s(\eta^{-1}(U_1)) \xrightarrow{(2)} H_0^s(\eta^{-1}(U_1)) \xrightarrow{(3)} H^s(\eta^{-1}(W_1))$$

where (1) is $\rho_1 \gamma \rightarrow (\rho_1 \gamma) \circ \eta$, (2) is

$\rho: \gamma \circ \eta \rightarrow (\rho_1 \gamma \circ \eta) \left(\frac{\partial y}{\partial x} \right) \left(\frac{\partial y}{\partial x} \right)$, and 3 is restriction to the set $\eta^{-1}(W_1)$. (1) is α_η which is continuous, (2) is a map of the form μ_e as described in corollary 31, so it is continuous. (3) is norm-decreasing so it is continuous. This shows that η^* is continuous from the $\|\cdot\|_1$ norm to the $\|\cdot\|_3$ norm. The proposition follows.

Now fix $\gamma \in \mathcal{M}$, (i.e. γ smooth) and define

$\psi: \mathcal{D}^{s+1} \rightarrow \mathcal{M}^s$ by $\psi(\eta) = A(\eta, \gamma)$.

Proposition 88: ψ is smooth.

Proof: Fix $\eta \in \mathcal{D}^{s+1}$, and smooth exponential map $e: T(M) \rightarrow M$. We look at ψ in terms of a chart at η ; i.e., consider $\psi: T_\eta \rightarrow \mathcal{D}^s$, ψ defined near zero in T_η . $V \rightarrow \psi(V)$ is $V \rightarrow (\gamma \circ e \circ V)(Te \circ V \cdot TV)(Te \circ V \cdot TV)$ where "o" means composition of maps and "." means multiplication of matrices or linear maps. Since γ and e are smooth, $V \rightarrow \gamma \circ e \circ V$ is smooth by example 48. Similarly $V \rightarrow Te \circ V$ is smooth. Also $V \rightarrow TV$ is clearly a continuous linear map from $H^{s+1}(U)$ to $H^s(U)$. All multiplications are smooth by lemma 29. Therefore ψ is smooth. q.e.d.

Corollary 89: For all $\gamma \in \mathcal{M}^s$, ψ_γ is continuous.

Proof: Everything remains smooth as above except $V \rightarrow \gamma \circ e \circ V$. But since $e \circ V$ is a diffeomorphism, we can use the weak ω -lemma 34, as in corollary 55, to ascertain that $V \rightarrow \gamma \circ e \circ V$ is continuous. Therefore ψ_γ is continuous. q.e.d.

Now we discuss the subgroup I_γ of \mathcal{D}^{s+1} which is the isotropy group of the action A , and give a manifold structure to the coset space $\mathcal{D}^{s+1}/I_\gamma$.

Definition 90: $I_\gamma = \{\eta \in \mathcal{D}^{s+1} \mid A(\eta, \gamma) = \gamma\}$.

From the fact that A is an action it is clear that I_γ is a subgroup of \mathcal{D}^{s+1} . Also we know that \mathcal{D}^{s+1} consists only of C^1 homeomorphisms. Therefore if $\eta \in I_\gamma$, η is a C^1 homeomorphism such that $\eta^*(\gamma) = \gamma$. This means that η is an isometry with respect to the metric γ . Clearly, any isometry of γ is in I_γ , so I_γ is the group of isometries of γ . The following results are well known:

Theorem 91: Let M be a smooth compact manifold with smooth Riemannian metric γ . Let $\{g_n\}$, g be in I_γ . If $g_n \rightarrow g$ and $Tg_n \rightarrow Tg$ uniformly on M (i.e., C^1 convergence), then $g_n \rightarrow g$ uniformly in all derivatives (i.e., C^∞). If I_γ is given the topology of uniform C^k convergence ($1 \leq k \leq \infty$), I_γ is a compact Lie group. By the first part of this theorem, the topology of I_γ is independent of k . Also, $A: I_\gamma \times M \rightarrow M$ defined by $A(g,p) = g(p)$ induces a natural identification, i , between the Lie algebra \mathfrak{g} of I_γ and the set \mathcal{A} of vector fields on M whose one parameter groups of diffeomorphisms lie in I_γ . i is defined by $i(X)_p = T_{(Id,p)}A(X,0)$ where Id is the identity of I_γ (see 12, 18).

We shall now show that the inclusion $I_\gamma \subseteq \mathcal{D}^s$ is a smooth map. We shall use the following form of the existence

theorem for differential equations:

Lemma 92: Let J be an open interval of \mathbb{R} containing zero, Y and U be open sets of some Banach spaces E and F , where $y_0 \in Y$, $u_0 \in U$, and let $f: J \times Y \times U \rightarrow F$ be a smooth map. Then there are subsets J_0, Y_0, U_0 of J, Y, U containing $0, y_0, u_0$ such that there is a smooth map $h: J_0 \times Y_0 \times U_0 \rightarrow U$, and

$$D_{(t,y,u)} h(1,0,0) = f(t,y,h(t,y,u)), \text{ for all } (t,y,u) \text{ in } J_0 \times Y_0 \times U_0.$$

Proof: See 13, p. 94.

Proposition 93: $i: I_\gamma \subseteq \mathfrak{D}^s$ is smooth.

Proof: We need only show smoothness at the identity Id , for, at any $\eta \in I_\gamma$, $i = R_{\eta^{-1}} \circ i \circ R_\eta$, and we know that multiplication on the right is smooth for \mathfrak{D}^s and I_γ .

To show smoothness at Id , we look at charts at Id in I_γ and \mathfrak{D}^s . Since I_γ is a Lie group, we know that the exponential map $\exp: \mathfrak{g} \rightarrow I_\gamma$ is a diffeomorphism from a neighborhood Y of zero in \mathfrak{g} to a neighborhood of Id in I_γ . We let (\exp, Y) give a chart for I_γ . As usual, a chart for \mathfrak{D}^s consists of a neighborhood (call it Z) of zero in T_{Id} . As a map from Y to Z

i has the following form: Pick $X \in Y$. Then X , as an element of \mathcal{O} , is a vector field on M . It generates a one parameter group of diffeomorphism η_t on M . If $E: T(M) \rightarrow M \times M$ is the fixed exponential on M , $i(X)$ is the vector field $p \rightarrow E^{-1}(p, \eta_1(p))$. We examine this situation in local coordinates. Take U a normal coordinate neighborhood at p in M (with respect to E), and let $\psi: T(U) \rightarrow U \times \mathbb{R}^n$ be the local trivialization of $T(M)$ induced by the coordinates. Assume Y small enough so that $f: \mathbb{R} \times Y \times U \rightarrow \mathbb{R}^n$ defined by $f(t, x, q) = \pi_2 \circ \psi \circ X(q)$ has range in U . Then the diffeomorphism $\eta(\lambda \eta_1)$ which corresponds to X is locally the map $q \rightarrow h(1, X, q)$ where h is defined by the above lemma (y_0 being the zero vector field).

$h: J_0 \times Y_0 \times U_0 \rightarrow U$ and since $f(t, \lambda X, q) = \lambda f(t, X, q)$ it is clear that $h(t, X, q) = h(1, tX, q)$ when both sides are defined. Therefore, by shrinking Y_0 , we can insure that J_0 is big enough to include 1 .

Define $\tilde{h}: Y_0 \rightarrow H^S(U_0, U)$ by $\tilde{h}(X)(u) = h(1, X, u)$ since h is smooth, all maps in the range of \tilde{h} are C^∞ and therefore H^S . Then $X \rightarrow i(X)$ is locally the following: $X \xrightarrow{(1)} \tilde{h}(X) \xrightarrow{(2)} (p \rightarrow (p, \tilde{h}(X)(p))) \xrightarrow{(3)} (p \rightarrow E^{-1}(p, \tilde{h}(X)(p)))$

The smoothness of this sequence is a particular case of the following lemma:

Lemma 94: Let A be a linear subspace of $C^\infty(T(M))$ which is a Banach space, and let $\phi: A \rightarrow \mathcal{D}^{s+1}$ be defined by $\phi(x) = \eta_1$, where $\{\eta_t\}$ is the one parameter family of diffeomorphisms corresponding to X . Then ϕ is smooth near zero in A .

Proof: We pick charts U_0, W_1 on M and trivializations ψ_j as usual. However, we insist that U_1 is small enough so that there is another chart U_1' such that $h: J_0 \times Y_0 \times U_1 \rightarrow U_1'$ (h, J_0, Y_0 as above). Let B be a neighborhood of zero in A such that for each of the Y_0 (one for each i) $B \subseteq Y_0$, such that if $X \in B$, then for each i , $\max_{u \in U_1} \|D_u(\pi_2 \circ \psi_j \circ X)\| \leq \varepsilon < 1$ and such that $\phi(B) \subseteq \psi_{Id}(\Theta_{Id})$ (the chart of \mathcal{D}^s about Id).

Then $\psi_{Id}^{-1} \circ \phi: B \rightarrow T_{Id}$ is locally the sequence (1), (2), (3) above.

We shall show that $\psi_{Id}^{-1} \circ \phi$ is continuous in the usual norms on each $H^s(W_1)$.

Then assuming $\psi_{Id}^{-1} \circ \phi$ is C^{k-1} we shall show that k^{th} Gateaux derivative converges locally and that it defines the following map from B to $L^k(A, T_{Id})$.

$$(+)\ d^k(\psi_{Id}^{-1} \circ \phi)(b, v^1 \dots v^k)(p) = D_p^k(\psi_{Id}^{-1} \circ \phi(b))(v^1(p), \dots, v^k(p))$$

where $\{v^1\} \subseteq A$, $b \in B$, $p \in M$. $\psi_{\text{Id}}^{-1} \circ \phi(b): M \rightarrow T(M)$
 so $D_p^k(\psi_{\text{Id}}^{-1} \circ \phi(b))$ is defined using local coordinates
 near p and $\psi_{\text{Id}}^{-1} \circ \phi(b)(p)$. Since $\psi_{\text{Id}}^{-1} \circ \phi(b)$ is smooth,
 this map has range in $L^k(A, T_{\text{Id}})$. That is, the first s
 derivatives of $d^k(\psi_{\text{Id}}^{-1} \circ \phi)(b, v^1 \dots v^k)$ are uniformly
 bounded by those of $\{v^1\}$ and of $\psi_{\text{Id}}^{-1} \circ \phi(b)$.

That the map is continuous will follow from the
 proof of the continuity of $\psi_{\text{Id}}^{-1} \circ \phi$ which we now give.

We work locally on U_1 and identify X with $\pi_2 \circ \psi_j \circ X$.
 Then on $H^s(W_1)$ we get the map $X \rightarrow \tilde{h}(X) \mapsto W_1$ from
 $Y_0 \rightarrow H^s(U_1, U'_1) \rightarrow H^s(W_1)$. By the existence theorem of
 differential equations,

$$(*) \quad h(1, X, p) = p + \int_0^1 X_h(t, x, p) dt.$$

We claim that if $X_n \rightarrow X$, C^∞ then $\tilde{h}(X_n) \rightarrow \tilde{h}(X)$
 C^∞ and therefore in $H^s(U_1, U'_1)$ as well.

$$\begin{aligned} |h(1, X_n, p) - h(1, X, p)| &\leq \int_0^1 |X_{nh}(t, X_n, p) - X_h(t, X, p)| dt \\ &\leq \|X_n - X\|_0 + \|DX_n\|_0 \|h(t, X_n, p) - h(t, X, p)\|_0 \end{aligned}$$

where $\|\cdot\|_0$ is C^0 norm on U'_1 .

But $\|DX_n\|_0 \leq \varepsilon < 1$ by the selection of B .

$$\begin{aligned} \text{Therefore, } \| \tilde{h}(X) - \tilde{h}(X_n) \|_0 &\leq \| X_n - X \|_0 / (1 - \| DX_n \|_0) \\ &\leq \| X_n - X \|_0 / (1 - \varepsilon) . \end{aligned}$$

$$\text{Therefore } \tilde{h}(X) \rightarrow \tilde{h}(X_n) \quad C^0 .$$

$$\text{From } (*) \quad D_{3(1,x,p)}^k h = D^k(\text{Id}) + \int_0^1 D_{3(1,x,p)}^k (X \circ h)$$

where D_3^k means the k^{th} derivative in the third component (i.e., $D_3 h(v) = Dh(0, 0, v)$) .

$$D_{3(1,x,p)}^k (X \circ h) = DX \cdot D_3^k h + \text{terms with } D_3^\ell h, \quad \ell < k .$$

$$\text{Therefore if we assume } \tilde{h}(X_n) \rightarrow \tilde{h}(X) \quad C^{k-1}$$

$$\begin{aligned} &|D_{3(1,x,p)}^k (X \circ h) - D_{3(1,x_n,p)}^k (X_n \circ h)| \\ &\leq \| DX - DX_n \|_0 |D_{3(1,x,p)}^k h| + \| DX_n \|_0 |D_{3(1,x,p)}^k h - D_{3(1,x_n,p)}^k h| \\ &\quad + \varepsilon_n \quad \text{where } \varepsilon_n \rightarrow 0 . \end{aligned}$$

$$\text{Therefore } \tilde{h}(X_n) \rightarrow \tilde{h}(X) \quad \text{in } C^k, \text{ as in the } C^0 \text{ case.}$$

This tells us that $\tilde{h}(X_n) \rightarrow \tilde{h}(X) \quad C^\infty$ and hence H^S .

Therefore $\psi_{\text{Id}}^{-1} \circ \phi$ is continuous. Also if $b_n \rightarrow b$ in B
 $D^k(\psi_{\text{Id}}^{-1} \circ \phi(b_n)) \rightarrow D^k(\psi_{\text{Id}}^{-1} \circ \phi(b))$ for all k , so the map
 $B \rightarrow L^k(A, T_{\text{Id}})$ defined above is continuous.

Now it remains to show the existence of
 $d^k(\psi_{\text{Id}}^{-1} \circ \phi)(b, v^1 \cdots v^k)$, and the equality at (+) .

We note that

$$d(\tilde{h})(X, V)(p) = \lim_{r \rightarrow 0} \left\{ \frac{1}{r} (h(1, X+rV, p) - h(1, X, p)) \right\} = D_3(1, X, p)h(V)$$

$$\begin{aligned} \text{By } (*) \quad d(\tilde{h})(X, V)(p) &= \lim_{p \rightarrow 0} \int_0^1 \frac{1}{r} (X_h(t, x+rV, p) \\ &\quad - X_h(t, x, p) + rV_h(t, x+rV, p)) dt . \end{aligned}$$

At each p , this converges to

$$\int_0^1 DX \cdot D_2(t, x, p)^{h(V)} + V_h(t, x, p) dt .$$

We will show that the convergence is uniform in p in all derivatives. We know from above that $\tilde{h}(X+rV) \rightarrow \tilde{h}(X)$ C^∞ so $V_h(t, X+rV, p) \rightarrow V_h(t, x, p)$ C^∞ in p . Hence

$$\begin{aligned} &\frac{1}{r} (h(1, X+rV, p) - h(1, X, p)) - D_2(1, x, p)^{h(V)} \\ &= \int_0^1 \frac{1}{r} (X_h(t, x+rV, p) - X_h(t, x, p)) - DX \cdot D_2(t, x, p)^{h(V)} dt + \varepsilon_r , \\ &\quad \varepsilon_r \rightarrow 0 \end{aligned}$$

$$\leq \| DX \|_0 \left\| \frac{1}{r} (h(t, X+rV, p) - h(t, X, p)) - D_2(t, x, p)^{h(V)} \right\|_0 + \varepsilon_r .$$

Now C^0 convergence follows since $\| DX \|_0 \leq \varepsilon < 1$.

C^∞ convergence follows as before, just differentiate everything with respect to p .

Assuming $d^{k-1}(\tilde{h})(X, V^1 \dots V^{k-1})$ we get $d^k(\tilde{h})(X, V^1 \dots V^k)$ by the same method and (+) holds. This proves the lemma

and also shows that $i: I_\gamma \rightarrow \mathcal{D}^S$ is smooth.

Definition 95: Let X and Y be smooth infinite dimensional manifolds, $f: X \rightarrow Y$ a smooth map. If at each $p \in X$, $T_p f: T_p(X) \rightarrow T_{f(p)}(Y)$ is injective, $T_p f(T_p(X))$ is closed in $T_{f(p)}(Y)$, and has a closed linear complement, then f is called an immersion. If, in addition, f is injective and is a homeomorphism into, then f is called an embedding and $f(X)$ is called a submanifold of Y (see 13, pp. 19-21).

Remark: If Y is a manifold based on a Hilbert space, any closed linear subspace of $T_{f(p)}(Y)$ has a closed complement, so f is an immersion if $T_p f$ is injective and $T_p f(T_p(X))$ is closed in $T_{f(p)}(Y)$.

Proposition 96: $i: I_\gamma \rightarrow \mathcal{D}^S$ is an embedding.

Proof: i is a smooth injective map and I_γ is compact, so i must be a homeomorphism onto a closed subset of \mathcal{D}^S . Therefore, we must show that $T_p i$ is injective and that $T_p i(T_p(I_\gamma))$ is closed in $T_{i(p)}(\mathcal{D}^S)$. We work at the identity in I_γ . Let $X \in \mathfrak{V}$, and let η_t be a one parameter family of diffeomorphisms generated by X . $t \rightarrow \eta_t$ is a smooth homomorphism $h: \mathbb{R} \rightarrow \mathcal{D}^S$, and $Ti(X) = T_0 h(\frac{d}{dt})$. Therefore, if $Ti(X) = 0$,

$T_0 h(\frac{d}{dt}) = 0$. But then for any $t_0 \in \mathbb{R}$,
 $T_{t_0} h(\frac{d}{dt}) = T(R_{\eta_{t_0}}) \circ T i(X) = 0$. Hence h is a constant
 map, so for all $t \in \mathbb{R}$, $\eta_t = \text{Id}$. Therefore $X = 0$.
 Hence $T i$ is injective at the identity. For any $\eta \in I_\gamma$,
 $T_\eta i = T R_\eta \circ T_{\text{Id}} i \circ T R_\eta^{-1}$ so $T_\eta i$ injective everywhere.
 Also $T_p(I_\gamma)$ is finite dimensional. Therefore
 $T_p i(T_p(I_\gamma))$ is finite dimensional and hence closed in
 $T_{i(p)}(\mathcal{D}^S)$. q.e.d.

Remark: As a result of the above proposition, we can
 identify I_γ and $i(I_\gamma)$.

Proposition 97: Let $c: I_\gamma \times \mathcal{D}^S \rightarrow \mathcal{D}^S$ be defined as
 composition of mappings. Then c is smooth.

Proof: We show smoothness at $\text{Id} \times \text{Id} \in I_\gamma \times \mathcal{D}^S$. Using
 a single chart T_{Id} , c takes (f,g) into the map
 $p \rightarrow E^{-1}(p, e f e g(p))$. Note that f is smooth, so

$$\begin{aligned}
 D_{(f,g)} c(u,v) &= D_{(p, e f e g(p))} E^{-1}(0, D_{f e g(p)} e(u_{e g(p)} \\
 &\quad + D_{e g(p)} f \cdot D_{g(p)} e(v_{g(p)})))
 \end{aligned}$$

the right side of the above equation being continuous in
 the local norms as usual. Since u is smooth, the exist-
 ence of higher derivatives follows in the same way. Now

pick $\eta \in I_\gamma$, $\zeta \in \mathcal{D}^S$. Then since
 $c = L_\eta \circ R_\zeta \circ c \circ (L_\eta^{-1}, R_\zeta^{-1}): I_\gamma \times \mathcal{D}^S \rightarrow \mathcal{D}^S$, the
 fact that c is smooth at $\text{Id} \times \text{Id}$ and the L_η and R_ζ
 are smooth (propositions 53 and 54) implies that c is
 smooth at (η, ζ) . q.e.d.

Corollary 98: Let $S = \bigcup_{\eta \in \mathcal{D}^S} T_{\text{Id}} R_\eta (T_{\text{Id}}(I_\gamma)) = T_{\text{Id}} R_\eta (\mathcal{G})$.

Then S is a smooth subbundle of $T(\mathcal{D}^S)$

Proof: It is clear that the fibres of S are closed
 (actually finite dimensional) linear subspaces of $T(\mathcal{D}^S)$
 Therefore, we need only construct smooth local trivializa-
 tions. Let U be a chart about the identity Id in \mathcal{D}^S .
 The chart map gives a trivialization $\psi: T(U) \cong U \times T_{\text{Id}}(\mathcal{D}^S)$
 which defines an identification of $T_u(\mathcal{D}^S)$ and $T_{\text{Id}}(\mathcal{D}^S)$, $u \in U$.

Let $a: U \times \mathcal{G} \rightarrow T(U)$ by $a(u, X) = T_{\text{Id}} R_u(X)$,
 $a(u, X) = T_{(\text{Id}, u)} c(X, 0)$, so a is smooth.
 $\psi \circ a: U \times \mathcal{G} \rightarrow U \times T_{\text{Id}}(\mathcal{D}^S)$ and it gives a smooth map
 $\tilde{\phi}: U \rightarrow L(\mathcal{G}, T_{\text{Id}}(\mathcal{D}^S))$ defined by $\tilde{\phi}(u)(X) = \psi \circ a(u, X)$.
 Let C be a complement to \mathcal{G} in $T_{\text{Id}}(\mathcal{D}^S)$ and extend
 $\tilde{\phi}$ to a map $\phi: U \rightarrow L(T_{\text{Id}}(\mathcal{D}^S), T_{\text{Id}}(\mathcal{D}^S))$ by
 $\phi(u)(X, Y) = \tilde{\phi}(u)(X) + Y$, $X \in \mathcal{G}$, $Y \in C$. ϕ is smooth
 and $\phi(\text{Id})$ is the identity map on $T_{\text{Id}}(\mathcal{D}^S)$. Therefore

by shrinking U , we can insure that the range of ϕ contains only linear isomorphisms. Let $\psi': T(U) \rightarrow U \times T_{\text{Id}}(\mathcal{D}^S)$ by $\psi'(V) = (\phi(u))^{-1} \circ \psi(V)$ $V \in T(U)$ and $u = \pi(V)$.

If $V \in S$, say $V \in T_{\text{Id}}^{R_u}(\mathcal{G})$, $\psi'(V) = (T_{\text{Id}}^{R_u})^{-1}(V) \in \mathcal{G}$

Therefore we get the commutative diagram:

$$\begin{array}{ccc} T(U) & \xrightarrow{\psi'} & U \times T_{\text{Id}}(\mathcal{D}^S) \\ U| & & U| \\ S(U) & \xrightarrow{\psi' \upharpoonright S(U)} & U \times \mathcal{G} \end{array}$$

If η is any point of \mathcal{D}^{S+1} , applying $T_{\text{Id}}(R_\eta)$ we get a commutative diagram:

$$\begin{array}{ccc} T(R_\eta(U)) & \longrightarrow & R_\eta(U) \times T_\eta(\mathcal{D}^S) \\ U| & & U| \\ S(R_\eta(U)) & \longrightarrow & R_\eta(U) \times T_\eta(\mathcal{G}) \end{array}$$

This diagram gives us a smooth local trivialization of S induced from a trivialization of $T(\mathcal{D}^S)$. Therefore S is a smooth subbundle. q.e.d.

Remark: $T(\mathcal{D}^S)$ has a global trivialization

$T(\mathcal{D}^S) \rightarrow \mathcal{D}^S \times T_{\text{Id}}(\mathcal{D}^S)$ defined by $V \rightarrow T_\eta(R_\eta)^{-1}(V)$.

However, we do not know that this trivialization is smooth.

On the other hand, the proof of the above corollary shows that S does have a smooth global trivialization because if $X \in \mathfrak{g}$, $\mathfrak{g} \times \mathcal{D}^S \rightarrow S$ defined by $X, \eta \rightarrow T_{\text{Id}} R_\eta(X)$ is a smooth map.

Lemma 99: S is an involutive subbundle; that is, if X and Y are vector fields that lie in S , so does $[X, Y]$.

Proof: We use the usual argument for Lie algebras ---

Take X, Y elements of \mathfrak{g} and define vector fields V, W on \mathcal{D}^S by $V_\eta = T_{\text{Id}} R_\eta(X)$, $W_\eta = T_{\text{Id}} R_\eta(Y)$. Clearly $Z = [V, W]$ is the vector field $Z_\eta = T_{\text{Id}} R_\eta([X, Y])$. But vector fields of form V span the fibres of S . Therefore S is involutive. q.e.d.

Now we are ready to use the Frobenius theorem to construct the manifold \mathcal{D}^S / I_γ .

Theorem 100: Let X be any manifold, S an involutive subbundle of $T(X)$. Then for any $z \in X$, there is a neighborhood W of z and a diffeomorphism $\phi: U \times V \rightarrow W$ (U, V open neighborhoods of zero in Banach spaces) such that $\phi(0, 0) = z$, the composition $T_1(U \times V) \hookrightarrow T(U \times V) \xrightarrow{T\phi} T(W)$ is a bundle isomorphism onto $S(W)$, where $T_1(U \times V)$ is the subbundle of $T(U \times V)$ whose fibres are $T_x(U) \times 0 \subseteq T_x(U) \times T_y(V) = T_{(x,y)}(U, V)$

$$x \in U, y \in V.$$

Proof: See 13, pp. 91-96.

The elements of S are (from their definition) the vectors in $T(\mathcal{D}^S)$ which are tangent to some coset $I_\gamma \eta$; that is the elements of $T(I_\gamma \eta) \subseteq T(\mathcal{D}^S)$, (the inclusion comes from the fact that $I_\gamma \eta = R_\eta(I_\gamma)$ is a submanifold).

Therefore, applying the above theorem to S we get the following:

Proposition 101: For any $\eta \in \mathcal{D}^S$, there is a neighborhood W of η and a diffeomorphism $\phi: U \times V \rightarrow W$ such that for any fixed $v \in V$, $\phi|_{U \times \{v\}}$ is a diffeomorphism onto a neighborhood of $\phi(0, v)$ in the coset $I_\gamma \phi(0, v)$.

Let $\pi: \mathcal{D}^S \rightarrow \mathcal{D}^S/I_\gamma$ be the usual projection, and give \mathcal{D}^S/I_γ the quotient topology, so that π is continuous and open. We shall define a differential structure for \mathcal{D}^S/I_γ near the identity coset.

Pick W a neighborhood of Id so that $\phi: U \times V \rightarrow W$ as in the above proposition. Since $i: I_\gamma \hookrightarrow \mathcal{D}^S$ is a homeomorphism into, we can make W small enough so that $I_\gamma \cap W = \phi(U, 0)$. Clearly we can restrict W so that U, V and W are connected.

Lemma 102: There exists connected neighborhoods U_0, V_0 of

zero, included in U, V , such that if $W_0 = \phi(U_0 \times V_0)$, for any $\eta \in W_0$, there exists a $v \in V_0$ such that $I_\gamma \eta \cap W_0 = \phi(U_0, v)$.

Proof: Clearly $\phi(u_0, v) \subseteq I_\gamma \eta \cap W_0$ for some η in W_0 .

Since \mathcal{D}^S is a topological group, we can pick neighborhoods W_1, W_2 of Id such that $W_1^2 \subseteq W$, $W_2 W_2^{-1} \subseteq W_1$, and $W_1 = \phi(U_1 \times V_1)$, where U_1, V_1 are connected balls about zero included in U and V respectively.

Pick U_0, V_0 small enough so that $\phi(U_0 \times V_0) = W_0 \subseteq W_2$.

Pick $\eta \in W_0$, and let a be in $I_\gamma \eta \cap W_0$. Then $a\eta^{-1} \in I_\gamma \cap W_0 W_0^{-1} \subseteq I_\gamma \cap W_1$. Therefore $a\eta^{-1} \in \phi(U_1, 0)$.

Let ℓ be a line in U_1 from 0 to $\phi^{-1}(a\eta^{-1})$.

Then $\phi(\ell)$ is a smooth curve from Id to $a\eta^{-1}$ in

$I_\gamma \cap W_1$. Therefore $\phi(\ell)\eta$ is a smooth curve C from η

to a in $(I_\gamma \cap W_1)\eta \subseteq I_\gamma \eta \cap W \subseteq \phi(U \times V)$. From the

construction of ϕ , we know that the tangent to C at

any point lies in $T\phi(T(U))$. Therefore, if $\eta = \phi(u, v_0)$

and $a = \phi(u', v)$, $v_0 = v$, so $a, \eta \in \phi(U, v_0)$. Since

a was any element of $I_\gamma \eta \cap W_0$, $I_\gamma \eta \cap W_0 \subseteq \phi(U_0, v_0)$. q.e.d.

Remark: Because of this lemma, $\pi \circ \phi \upharpoonright (0 \times V_0): V_0 \rightarrow \mathcal{D}^S/I_\gamma$ is a homeomorphism onto a neighborhood of I_γ in \mathcal{D}^S/I_γ .

Definition 103: Give \mathcal{D}^S/I_γ a manifold structure by

declaring that for each element $I_\gamma \eta$ of \mathcal{D}^S/I_γ
 $\psi_\eta = \pi \circ R_\eta \circ \phi: V_0 \rightarrow \mathcal{D}^S/I_\gamma$ is a chart at $I_\gamma \eta$.

It is clear that ψ_η is injective and $\psi_\eta(V_0)$ covers a neighborhood of η , so we need only check that the charts are smoothly compatible. $\psi_\eta^{-1} \circ \psi_\xi$ (where defined) has form: $V \xrightarrow{(1)} \phi(0, V) \xrightarrow{(2)} \phi(0, V)\xi \xrightarrow{(3)} \phi(0, V)\xi\eta^{-1} \xrightarrow{(4)} I_\gamma \phi(0, V)\xi\eta^{-1} \xrightarrow{(5)} \pi_2 \phi^{-1}(I_\gamma \phi(0, V)\xi\eta^{-1})$ where $\pi_2: U_0 \times V_0 \rightarrow V_0$ is the projection on the second factor. The first three steps are clearly smooth; the last two comprise the map $V \rightarrow \pi_2 \circ \phi^{-1}(\phi(0, V)\xi\eta^{-1})$ which is also smooth. Therefore \mathcal{D}^S/I_γ is a manifold.

Remark: R_η acting on \mathcal{D}^S/I_γ in the usual way is smooth.

Proposition 104: The map $\pi: \mathcal{D}^S \rightarrow \mathcal{D}^S/I_\gamma$ admits a smooth local cross section at any coset $I_\gamma \eta$.

Proof: Let $\chi_\eta: \mathcal{D}^S/I_\gamma \rightarrow \mathcal{D}^S$ be defined in a neighborhood of η by $\chi_\eta = R_\eta \circ \phi \circ \psi_\eta^{-1}$. χ_η is a smooth map $\chi_\eta(I_\gamma \eta) = \eta$, and $\pi \circ \chi_\eta$ is the identity map.

Corollary 105: A map $f: \mathcal{D}^S/I_\gamma \rightarrow N$ (N any manifold) is smooth if and only if $f \circ \pi: \mathcal{D}^S \rightarrow N$ is smooth. In particular π is smooth.

Proof: Assume $f \circ \pi$ is smooth. Near a coset $I_\gamma \eta$
 $f = f \circ \pi \circ \chi_\eta$. $f \circ \pi$ and χ_η are smooth so f is.
 π is smooth because, using charts $R_\eta \circ \phi$ and ψ_η , π
becomes $\pi_2: U \times V \rightarrow V$. Hence if f is smooth $\pi \circ f$
is smooth. q.e.d.

VI. The Map of \mathcal{D}^{s+1} onto an Orbit through γ .

Now let $\psi_\gamma: \mathcal{D}^{s+1} \rightarrow \mathcal{M}^s$ (also called ψ) be defined as before by $\psi_\gamma(\eta) = \eta^*(\gamma)$. Since $\psi_\gamma(I_\gamma) = \gamma$, ψ_γ defines a map $\phi_\gamma: \mathcal{D}^{s+1}/I_\gamma \rightarrow \mathcal{M}^s$ (also called ϕ) by $\phi(I_\gamma \eta) = \eta^*(\gamma)$. If $\eta' = a\eta$ ($a \in I_\gamma$) then $(a\eta)^*(\gamma) = \eta^*a^*(\gamma) = \eta^*(\gamma)$, so the definition of ϕ_γ makes sense.

Lemma 106: ϕ is smooth and injective.

Proof: ψ is smooth by Proposition 88 and clearly $\phi \circ \pi = \psi$. Therefore by the Corollary 105, ϕ is smooth. $\phi(I_\gamma \eta) = \phi(I_\gamma \xi)$ implies $\eta^*(\gamma) = \xi^*(\gamma)$ so $\eta\xi^{-1} \in I_\gamma$ and $I_\gamma \eta = I_\gamma \xi$. q.e.d.

We are interested in showing that ϕ is an immersion. To do so we must show that $T_p \phi: T_p(\mathcal{D}^{s+1}/I_\gamma) \rightarrow T_{\phi(p)}(\mathcal{M}^s)$ is injective and has closed range. We shall first study the map ψ , and in particular $T_{\text{Id}} \psi: T_{\text{Id}}(\mathcal{D}^{s+1}) \rightarrow T_\gamma(\mathcal{M}^s)$. We recall that $T_{\text{Id}}(\mathcal{D}^{s+1})$ is naturally identified with $H^{s+1}(T(M))$ and $T_\gamma(\mathcal{M}^s)$ is identified with $H^s(S^2 T^*)$.

Lemma 107: With the above identification,

$T_{\text{Id}} \psi: H^{s+1}(T(M)) \rightarrow H^s(S^2 T^*)$ is the differential

operator α defined by $\alpha(x) = \theta_x(\gamma)$, (the Lie derivative of γ with respect to the vector field x). (See Example 67).

Proof: We know that α and $T_{\text{Id}}\psi$ are both continuous linear maps from $H^{s+1}(T(M))$ to $H^s(S^2T^*)$. Therefore it is enough to show that they agree on the dense subset $C^\infty(T(M))$ of $H^s(T(M))$. Pick $V \in C^\infty(T(M))$. Then V generates a one parameter group of diffeomorphisms $\{\eta_t\}$. We first point out that the map $C: \mathbb{R} \rightarrow \mathcal{D}^{s+1}$ by $t \rightarrow \eta_t$ is a smooth curve in \mathcal{D}^{s+1} whose tangent at any point t_0 is the map $p \rightarrow V_{\eta_{t_0}}(p)$, an element of $T_{\eta_{t_0}}$. Smoothness near Id is shown in Lemma 94. Smoothness everywhere then follows from the fact that right multiplication is smooth in \mathcal{D}^s and \mathbb{R} .

From this it is also clear that the tangent to C at $C(t)$ is $V \circ \eta_t \in T_{\eta_t}(\mathcal{D}^{s+1})$.

Now to compute $T_{\text{Id}}\psi(V)$ we look at the tangent to the curve $\psi \circ C(t)$, at zero,

$$\frac{d}{dt} (\psi \circ C(t))|_0 = \frac{d}{dt} (\eta_t^*(\gamma))|_0.$$

At each point $p \in M$, $\frac{d}{dt} (\eta_t^*(\gamma_p))|_0$ is by definition $\theta_V(\gamma)_p$. Therefore since ψ is smooth, $\frac{d}{dt} (\psi \circ C(t))$ exists in $H^s(S^2T^*)$ so it must be the map $p \rightarrow \theta_V(\gamma)_p$ which is $\alpha(V)$. The lemma follows.

We now look at $T_\eta\psi$ for any $\eta \in \mathcal{D}^{s+1}$. We

recall that $\eta^*: \mathcal{M}^S \rightarrow \mathcal{M}^S$ is a restriction of the continuous linear map $\eta^*: H^S(S^2T^*) \rightarrow H^S(S^2T^*)$, so for any $\lambda \in \mathcal{M}^S$, $T_\lambda \eta^* = \eta^*: H^S(S^2T^*) \rightarrow H^S(S^2T^*)$.

Proposition 108: $T_\eta \psi(T_\eta(\mathcal{D}^{S+1})) = \eta^* \circ \alpha \circ T_\eta(R_{\eta^{-1}})(T_\eta(\mathcal{D}^{S+1}))$.

Proof: $\psi = \eta^* \circ \psi \circ R_{\eta^{-1}}$ for $\eta^* \circ \psi \circ R_{\eta^{-1}}(\xi) = \eta^* \psi(\xi \eta^{-1}) = \eta^*(\xi \eta^{-1})^*(\gamma) = \xi^*(\gamma) = \psi(\xi)$. Therefore $T_\eta \psi = T_\gamma \eta^* \circ T_{\text{Id}} \psi \circ T_\eta(R_{\eta^{-1}})$ so $T_\eta \psi(T_\eta(\mathcal{D}^{S+1})) = \eta^* \circ \alpha \circ T_\eta(R_{\eta^{-1}})(T_\eta(\mathcal{D}^{S+1}))$.

Corollary 109: $T_\eta \psi(T_\eta(\mathcal{D}^{S+1}))$ and $T_{\text{Id}} \psi(T_{\text{Id}}(\mathcal{D}^{S+1}))$ are isomorphic subspaces of $H^S(S^2T^*)$.

Proof: η^* and $T_\eta(R_{\eta^{-1}})$ are linear isomorphisms.

Our next goal is to show that $T_\eta \psi(T_\eta(\mathcal{D}^{S+1}))$ is closed in $T_\psi(\eta)(\mathcal{M}^S) = H^S(S^2T^*)$. Because of the above corollary it is sufficient to do this for the case $\eta = \text{Id}$. We know $T_{\text{Id}} \psi = \alpha: H^{S+1}(T(M)) \rightarrow H^S(S^2T^*)$, so we examine α . To do this we must use a few properties of differential operators which we now explain.

Let E and F be vector bundles on M and let $D: C^\infty(E) \rightarrow C^\infty(F)$ be a k th order differential operator. Then as we know, D induces a linear

bundle map: $\tilde{D}: J^k(E) \rightarrow F$. Also we have the exact sequence: $0 \rightarrow S^k T^* \times E \xrightarrow{i_k} J^k(E) \rightarrow J^{k-1}(E) \rightarrow 0$.

Definition 110: For any $p \in M$ and $t \in T_p^*(M)$, there is a linear map $\sigma_t(D): E_p \rightarrow F_p$ defined by $\sigma_t(D)(e) = D \circ i_k(t \times t \cdots \times t \times e)$, which is called the symbol of D at t . D is called elliptic if $\sigma_t(D)$ is bijective for all non-zero t in $T^*(M)$.

Let E and F have smooth inner products $(\cdot, \cdot)_E$ and $(\cdot, \cdot)_F$ and let ρ be a smooth measure for M so that $H^0(E)$ and $H^0(F)$ have explicit inner products.

Definition 111: A k th order differential operator $D^*: C^\infty(F) \rightarrow C^\infty(E)$ is an adjoint of D if for all $e \in C^\infty(E)$, $f \in C^\infty(F)$, $\int_M (De, f)_F d\rho = \int_M (e, D^*f)_E d\rho$.

Proposition 112: Every operator D has a unique adjoint which we call D^* .

Proof: See 17, pp. 70-72.

Note: $\int_M (De, f) = \int (e, D^*f)$ for all $e \in H^S(E)$, $f \in H^{S-k}(F)$ if $s \geq k$. This is true because the formula holds on the dense subset of smooth sections, and D and D^* are continuous linear maps.

Proposition 113: For all $t \in T^*(M)$, $\sigma_t(D^* \circ D) = \sigma_t(D^*) \circ \sigma_t(D)$ and $\sigma_t(D^*) = (\sigma_t(D))^*$ where $(\sigma_t(D))^*: F_p \rightarrow E_p$ is the adjoint of the linear map $\sigma_t(D): E_p \rightarrow F_p$ with respect to the inner products on E_p and F_p .

Proof: See 17, pp. 68 and 73.

Corollary 114: If for all non-zero $t \in T^*(M)$, $\sigma_t(D): E_p \rightarrow F_p$ is injective, then $D^* \circ D: C^\infty(E) \rightarrow C^\infty(E)$ is an elliptic operator.

Proof: Any injective linear map followed by its adjoint is an isomorphism. Apply this fact to $\sigma_t(D)$.

Since $\int_M (D^* \circ D e, e') = \int_M (D e, D e') = \int_M (e, D^* \circ D e') ,$
 $D^* \circ D$ is its own adjoint. Let $D_S: H^S(E) \rightarrow H^{S-k}(F)$
be the extension of $D: C^\infty(E) \rightarrow C^\infty(F)$.

Proposition 115: If D is a k th order elliptic operator from E to F then

1) $\text{Ker } D = \text{Ker } D_S$ is a finite dimensional subspace of $C^\infty(E)$ and similarly $\text{Ker } D^* = \text{Ker } D_S^*$, a finite dimensional subspace of $C^\infty(F)$.

2) $H^{S-k}(F) = \text{im } D_S \oplus \text{Ker } D^*$, in particular $\text{im } D_S$ is closed in $H^{S-k}(F)$.

Proof: See 17, 178-179.

Lemma 116: Let A, B be Banach spaces $f: A \rightarrow B$ a continuous linear map. Assume $C \subset B$ is an (algebraic) linear complement to $f(A)$ and C is closed in B . Then $f(A)$ closed in B , and $B = f(A) \oplus C$.

Proof: See 17, p. 119.

Now let $D: C^\infty(E) \rightarrow C^\infty(F)$ have the property that for all non-zero t in $T^*(M)$, $\sigma_t(D)$ is injective. Then $D^* \circ D$ is a self-adjoint elliptic operator.

Proposition 117: For $s \geq k$, $H^{s-k}(E) = \text{im } (D^* \circ D)_{s+k} \oplus \text{Ker } D^* \circ D$. Also $\text{Ker } D^* \circ D = \text{Ker } D$ and $\text{im } (D^* \circ D)_{s+k} = \text{im } D_s^*$.

Proof: The first statement follows from 2) of Proposition 115, and the fact $D^* \circ D$ is its own adjoint. Let $(\cdot, \cdot)_0$ be the inner product on $H^0(E)$ or $H^0(F)$. Then $(e, e')_0 = \int_M (e, e') dp$. Clearly $\text{Ker } D^* \circ D \supset \text{Ker } D$. Also if $D^* De = 0$, $0 = (D^* De, e)_0 = (De, De)_0$ so $De = 0$. Hence $\text{Ker } D^* \circ D = \text{Ker } D$. Furthermore it is clear that $\text{im } (D^* \circ D)_{s+k} \subset \text{im } D_s^*$ because $(D^* \circ D)_{s+k} = D_s^* \circ D_{s+k}$. Therefore in view of the first statement we need only show that $\text{im } D_s^* \cap \text{Ker } D^* \circ D = \{0\}$ or

that $\text{im } D_S^* \cap \text{Ker } D = \{0\}$. If $De = 0$ and $e = D^*f$ then $D \circ D^*f = 0$ so $(D \circ D^*f, f)_0 = 0$, or $e = D^*f = 0$. The proposition follows.

Corollary 118: $\text{im } D_{S+k}$ is closed in $H^S(F)$, and $H^S(F) = \text{im } D_{S+k} \oplus \text{Ker } D_S^*$.

Proof: Since $\text{Ker } D_S^*$ is closed we need only show that the above direct sum is true in the algebraic sense. Then by Lemma 116 it follows that $\text{im } D_{S+k}$ is closed and the sum is topological.

We first show that $\text{im } D_{S+k} \cap \text{Ker } D_S^* = \{0\}$ let $f \in H^S(F)$ such that $D_S^*f = 0$ and $f = D_{S+k}e$. Then $0 = (D_S^* \circ D_{S+k}e, e)_0 = (D_{S+k}e, D_{S+k}e)_0$ so $f = 0$. Now we show $\text{im } D_{S+k} + \text{Ker } D_S^*$ spans $H^S(F)$. $H^S(F) = D_S^{*-1}(D_S^*(H^S(F)))$, and $D_S^*(H^S(F)) = D_S^* \circ D_{S+k}(H^{S+k}(E))$ by the above proposition. Therefore

$$\begin{aligned} H^S(F) &= D_S^{*-1}(D_S^* \circ D_{S+k}(H^{S+k}(E))) \\ &= \text{Ker } D_S^* + D_{S+k}(H^{S+k}(E)) . \end{aligned}$$

The corollary follows.

Now we apply the above corollary to the first order operator α .

Proposition 119: Let $\alpha_{S+1}: H^{S+1}(T(M)) \rightarrow H^S(S^2T^*)$.

Then $\text{im } \alpha_{s+1}$ is closed and has closed complement in $H^s(S^2T^*)$.

Proof: All we need to show is that for any $p \in M$ if $t \in T_p^*(M)$ and $t \neq 0$, $\sigma_t(\alpha)$ is injective. Pick local coordinates $x^1 \dots x^n$ on a neighborhood U of p , and let $\gamma = \gamma_{ij} dx^i \otimes dx^j$. Let $f \in C^\infty(U, \mathbb{R})$ be such that $(df)_p = t$, $f(p) = 0$, let $V = v^i \frac{\partial}{\partial x^i}$ be a smooth vector field defined near p . Then

$$\begin{aligned} \sigma_t(\alpha)(V_p) &= \theta_{fV}(\gamma)_p \\ &= (fv^k \frac{\partial \gamma_{ij}}{\partial x^k} + \gamma_{ki} (f \frac{\partial v^k}{\partial x^j} + \frac{\partial f}{\partial x^j} v^k) \\ &\quad + \gamma_{kj} (f \frac{\partial v^k}{\partial x^i} + \frac{\partial f}{\partial x^i} v^k))_p dx^i \otimes dx^j. \end{aligned}$$

$$\text{But } f(p) = 0, \text{ so } \theta_{fV}(\gamma)_p = (\gamma_{ki} \frac{\partial f}{\partial x^j} v^k + \gamma_{kj} \frac{\partial f}{\partial x^i} v^k)_p dx^i \otimes dx^j$$

From this formula we see that if V^* is the element of $T_p^*(M)$ corresponding to V_p under the isomorphism induced by γ , $\theta_{fV}(\gamma)_p = t \otimes V^* + V^* \otimes t$. Therefore $\sigma_t(\alpha)(V_p) = 0$ implies $t = 0$ or $V_p = 0$. The proposition follows.

Now we know that for all $\eta \in \mathcal{D}^{s+1}$, the image of $T_\eta \psi$ is closed in $T_{\psi(\eta)}(\mathcal{M}^s)$. $\psi = \phi \circ \pi$ and $T_\eta \pi$ is clearly onto $T_{\pi(\eta)}(\mathcal{D}^{s+1}/I_\gamma)$, so the image of $T_\eta \psi$

is the same as that of $T_{\pi(\eta)}\phi$.

In order to show that ϕ is an immersion, we must also show that $T\phi$ is injective on each tangent space. Since at any $\eta \in \mathcal{D}^{s+1}$, $T_\eta\psi = T_{\pi(\eta)}\phi \circ T_\eta\pi$, it is enough to show the following:

Proposition 120: If $V \in T_\eta(\mathcal{D}^{s+1})$ and $T_\eta\psi(V) = 0$, then $T_\eta\pi(V) = 0$.

Proof: We first consider the case $\eta = \text{Id}$. Here we need to show $\alpha(V) = 0$ implies $T_{\text{Id}}\pi(V) = 0$. But if $\alpha(V) = 0$, V is a smooth vector field and $\Theta_V(\gamma) = 0$. Hence if $\{\eta_t\}$ is the one parameter group generated by V , $\{\eta_t\} \subset I_\gamma$. Therefore $\pi(\{\eta_t\})$ lies in the identity coset I_γ , so $T_{\text{Id}}\pi(V) = 0$. For any $\eta \in \mathcal{D}^{s+1}$, $T_\eta\psi = \eta^* \circ \alpha \circ T_\eta R_{\eta^{-1}}$. Therefore for $V \in T_\eta(\mathcal{D}^{s+1})$, $T_\eta\psi(V) = 0$ only if $T_\eta R_{\eta^{-1}}(V) \in \text{Ker } \alpha$, and this occurs only if $T_{\text{Id}}\pi \circ T_\eta R_{\eta^{-1}}(V) = 0$. Let $\tilde{R}_\eta: \mathcal{D}^{s+1}/I_\gamma \rightarrow \mathcal{D}^{s+1}/I_\gamma$ be defined by $I_{\gamma\xi} \xrightarrow{\eta} I_{\gamma\xi\eta}$. $\tilde{R}_\eta \circ \pi = \pi \circ R_\eta$. Therefore $\tilde{R}_\eta \circ \pi$ is smooth, so \tilde{R}_η is smooth. Also $\tilde{R}_\eta \circ \pi \circ R_{\eta^{-1}} = \pi$. Hence if $T_{\text{Id}}\pi \circ T_\eta R_{\eta^{-1}}(V) = 0$, $T_{\pi(\text{Id})}\tilde{R}_\eta \circ T_{\text{Id}}\pi \circ T_\eta R_{\eta^{-1}}(V) = 0$, so $T_\eta\pi(V) = 0$. q.e.d.

We now have shown:

Proposition 121: $\phi_\gamma: \mathcal{D}^{s+1}/I_\gamma \rightarrow \mathcal{M}^s$ is an injective immersion.

VII. The Group \mathcal{D}^{s+1} Acts Isometrically on \mathcal{M}^s .

For any $\eta \in \mathcal{D}^{s+1}$ we know $\eta^*: \mathcal{M}^s \rightarrow \mathcal{M}^s$ is a diffeomorphism with inverse $(\eta^{-1})^*$.

We shall show that η^* is an isometry --- that it preserves the Riemannian metric $\mu: \mathcal{M}^s \rightarrow B(H^s(S^2 T^*))$. This demonstration is a step by step examination of the construction of μ .

Assume η is smooth. Let $\langle \rangle$ and $\langle \rangle'$ be the inner products on $T(M)$ defined by γ and $\eta^*(\gamma)$ respectively, and let ∇, D and ∇', D' be respectively the corresponding affine connections on $T(M)$ and covariant derivatives on $T^*(M)$.

Lemma 122: If X, Y are vector fields on M ,

$$\nabla_X' Y = (T\eta)^{-1} \nabla_{T\eta X} T\eta Y.$$

Proof: ∇' is uniquely characterized by the two properties:

$$1) \quad X \langle Y, Y \rangle' = 2 \langle \nabla_X' Y, Y \rangle'$$

$$2) \quad \nabla_X' Y - \nabla_Y' X - [X, Y] = 0 \quad (\text{see 11, p. 71}).$$

We shall establish these properties for $T\eta^{-1} \nabla_{T\eta X} T\eta Y$ using the fact that analogous properties hold for ∇ .

$$1) \quad X_p \langle Y, Y \rangle' = X_p \langle T\eta Y, T\eta Y \rangle_{\eta(p)} = T\eta(X_p) (\langle T\eta Y, T\eta Y \rangle)$$

$$= 2 \langle \nabla_{T\eta X} T\eta Y, T\eta Y \rangle = 2 \langle T\eta^{-1} \nabla_{T\eta X} T\eta Y, Y \rangle'$$

$$2) \quad (T\eta)^{-1} \nabla_{T\eta X} T\eta Y - T\eta^{-1} \nabla_{T\eta Y} T\eta X - [X, Y]$$

$$= T\eta^{-1} (\nabla_{T\eta X} T\eta Y - \nabla_{T\eta Y} T\eta X - [T\eta X, T\eta Y])$$

$$= T\eta^{-1}(0) = 0.$$

The lemma follows.

Lemma 123: $\eta^* \circ D = D' \circ T\eta^*$ where η^* acts on $T^*(M) \otimes T^*(M)$ in the usual way.

Proof: Let ω be a 1-form on M and W the vector field which corresponds to ω by $\omega(Y) = \langle W, Y \rangle$; i.e., $W = \gamma^{-1}(\omega)$.

$D\omega$ is by definition the map $X \rightarrow \gamma(\nabla_X W)$. Therefore, $\eta^* D\omega$ is the map $X \rightarrow T\eta^*(\gamma(\nabla_{T\eta X} W))$ which is $X \rightarrow \eta^*(\gamma)(\nabla_X' T\eta^{-1} W)$. Also

$T\eta^*(\omega) = T\eta^*(\gamma(W)) = \eta^*(\gamma)(T\eta^{-1} W)$. Hence $D' \circ T\eta^*(\omega)$ is the map $X \rightarrow \eta^*(\gamma)(\nabla_X' T\eta^{-1} W)$. The lemma follows.

Corollary 124: If D_k, D_k' are the covariant derivatives on $(T^*)^k$ induced by γ and $\eta^*(\gamma)$ respectively, then

$\eta^* \circ D_k = D'_k \circ \eta^*$ where η^* acts on $(T^*)^k$ and $(T^*)^{k+1}$.

Proof: Follows directly from the definitions of D and D' and the fact that if $\tau \in (T^*)^k$, $\sigma \in (T^*)^\ell$,
 $\eta^*(\tau \otimes \sigma) = \eta^*(\tau) \otimes \eta^*(\sigma)$.

Let $\langle \cdot \rangle_k$ and $\langle \cdot \rangle'_k$ be the inner products on $(T^*)^k$ induced by γ and $\eta^*(\gamma)$ respectively.

Lemma 125: Let $\sigma, \tau \in (T^*)^k$. $\langle \sigma, \tau \rangle_k = \langle \eta^* \sigma, \eta^* \tau \rangle'_k$.

Proof: Use local coordinates: Let

$$\tau = \tau_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}, \quad \sigma = \sigma_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$$

$\gamma = \gamma_{ij} dx^i \otimes dx^j$, and η be the map

$$y^1(x^1 \dots x^n), \dots, y^n(x^1 \dots x^n).$$

$$\langle \sigma, \tau \rangle_k = \tau_{i_1 \dots i_k} \sigma_{j_1 \dots j_k} \gamma^{i_1 j_1} \dots \gamma^{i_k j_k} \text{ where } \gamma^{ij}$$

is the matrix inverse of γ_{ij} .

$$\langle \eta^*(\sigma), \eta^*(\tau) \rangle'_k = \frac{\partial y^{i_1}}{\partial x^{l_1}} \dots \frac{\partial y^{i_k}}{\partial x^{l_k}} \tau_{i_1 \dots i_k} \frac{\partial y^{j_1}}{\partial x^{m_1}} \dots \frac{\partial y^{j_k}}{\partial x^{m_k}} \sigma_{j_1 \dots j_k}$$

$$\eta^*(\gamma)^{l_1 m_1} \dots \eta^*(\gamma)^{l_k m_k}$$

$$\text{and } \eta^*(\gamma)^{lm} = (\gamma_{ij} \frac{\partial y^i}{\partial x^l} \frac{\partial y^j}{\partial x^m})^{-1}. \text{ The lemma follows.}$$

Corollary 126: Let $\langle \rangle$ and $\langle \rangle'$ be the inner products induced on $J^S(S^2T^*)$ by γ, D and $\eta^*(\gamma), D'$ respectively. Let g and h be smooth sections of S^2T^* and let \tilde{g}, \tilde{h} be the sections of $J^S(S^2T^*)$ induced by g, h , and $\widetilde{\eta^*(g)}, \widetilde{\eta^*(h)}$ be those induced by $\eta^*(g), \eta^*(h)$. Then $\langle \tilde{g}_{\eta(p)}, \tilde{h}_{\eta(p)} \rangle = \langle \widetilde{\eta^*(g)}_p, \widetilde{\eta^*(h)}_p \rangle'$.

Proof: This follows from the preceding lemma and corollary.

By the corollary $D'_k \circ D'_{k-1} \cdots D' \eta^*(g) = \eta^* \circ D_k \circ D_{k-1} \cdots D(g)$. Therefore $\widetilde{\eta^*(g)} = \eta^*(\tilde{g})$ where $\eta^*(\tilde{g})$ is defined by the effect of η^* on each component $S^{kT^*} \otimes S^{2T^*}$ and the isomorphism $J^S(S^2T^*) \cong \sum_{k=0}^S S^{kT^*} \otimes S^{2T^*}$ induced by γ and $\{D_k\}_{k=1}^S$. Now the corollary follows from the previous lemma.

Proposition 127: Let $\langle \rangle$ and $\langle \rangle'$ be the inner products $\mu(\gamma)$ and $\mu(\eta^*(\gamma))$ on $H^S(S^2T^*)$, and let g, h be smooth sections of S^2T^* . Then $(g, h) = (\eta^*(g), \eta^*(h))'$.

Proof: We use the previous corollary and examine the measures on M defined by γ and $\eta^*(\gamma)$.

Using local coordinates, let η be $y^1(x^1 \cdots x^n), \dots, y^n(x^1 \cdots x^n)$, $\gamma = \gamma_{1j}$. Then

$$(g, h) = \int \langle \tilde{g}, \tilde{h} \rangle_p (\det \gamma_{1j})_p^{1/2} dx^1 \cdots dx^n(p)$$

$$\begin{aligned}
 (\eta^*(g), \eta^*(h)) &= \int \langle \widetilde{\eta^*(g)}, \widetilde{\eta^*(h)} \rangle_p' \left(\det \left(\frac{\partial y^i}{\partial x^k} \gamma_{ij} \frac{\partial y^j}{\partial x^l} \right)_p \right)^{1/2} dx^1 \dots dx^n(p) \\
 &= \int \langle \widetilde{g}, \widetilde{h} \rangle_{\eta^{-1}(p)} \det \left(\frac{\partial y^k}{\partial x^l} \right)_p \left(\det \gamma_{ij} \right)_p^{1/2} dx^1 \dots dx^n(p) \\
 &= \int \langle \widetilde{g}, \widetilde{h} \rangle_p \left(\det \gamma_{ij} \right)_p^{1/2} dx^1 \dots dx^n(p)
 \end{aligned}$$

by the change of variables theorem. But this last term is (g, h) . q.e.d.

Corollary 128: $(g, h) = (\eta^*(g), \eta^*(h))'$ for $g, h \in H^S(S^{2T*})$.

Proof: $(\)$ and $(\)'$ are continuous and by the proposition, the equality holds on a dense subset of $H^S(S^{2T*})$.

Corollary 129: $(g, h) = (\eta^*(g), \eta^*(h))'$ for $g, h \in H^S(S^{2T*})$ and $\eta \in \mathcal{D}^{s+1}$.

Proof: Let $\eta_n \rightarrow \eta$ in \mathcal{D}^{s+1} $\{\eta_n\} \subseteq \mathcal{D}$. Let $(\)^n$ be the inner product $\mu(\eta_n^*(\gamma))$ on $H^S(S^{2T*})$. Since ψ and μ are continuous, $(\eta^*(g), \eta^*(h))^n \rightarrow (\eta^*(g), \eta^*(h))'$. Also $(\eta^*(g), \eta^*(h))^n = ((\eta_n^{-1})^* \circ \eta^*(g), (\eta_n^{-1})^* \circ \eta^*(h))$. Since ψ is smooth, and $(\eta \eta_n^{-1})^* = T\eta \eta_n^{-1}(\psi)$
 $(\eta_n^{-1})^* \eta^*(g) = (\eta \eta_n^{-1})^*(g) \rightarrow g$ and $(\eta \eta_n^{-1})^*(h) \rightarrow h$.

Therefore $(g, h) = (\eta^*(g), \eta^*(h))'$ q.e.d.

This finishes the demonstration that η^* is an isometry on \mathcal{M}^S .

\mathcal{M}^S is a connected manifold, so the Riemannian metric on \mathcal{M}^S defines a metric on \mathcal{M}^S in the usual way. (The distance between p and q is the minimum of the lengths of all curves from p to q .) This metric induces the given topology on \mathcal{M}^S , (see 19, p. 158), and any (Riemannian) isometry preserves the metric because it preserves the length of curves.

We have shown that $A: \mathcal{D}^{S+1} \times \mathcal{M}^S \rightarrow \mathcal{M}^S$ is continuous in each variable separately. We are now in a position to show joint continuity.

Proposition 130: A is jointly continuous in its two arguments.

Proof: Let $\gamma_n \rightarrow \gamma$ and $\eta_n \rightarrow \eta$ in \mathcal{M}^S and \mathcal{D}^{S+1} respectively.

$A(\eta_n, \gamma_n) = \eta_n^*(\gamma_n)$. $\eta_n^*(\gamma) \rightarrow \eta^*(\gamma) = A(\eta, \gamma)$ since ψ_γ continuous. If ρ_S is the metric on \mathcal{M}^S , then $\rho_S(\gamma_n, \gamma) \rightarrow 0$. Therefore, since η_n^* is an isometry,

$$\rho_S(\eta_n^*(\gamma_n), \eta_n^*(\gamma)) \rightarrow 0. \text{ Therefore } \eta_n^*(\gamma_n) \rightarrow \eta^*(\gamma),$$

so A is continuous. q.e.d.

We shall eventually look at the normal bundle ν on $\mathcal{D}^{s+1}/I_\gamma$ induced by the immersion ϕ and at the effect of the exponential map of \mathcal{M}^s on this bundle. First, however, we want to show that the exponential map commutes with the action of \mathcal{D}^{s+1} . Specifically we shall show:

Proposition 131: If $f: \mathcal{M}^s \rightarrow \mathcal{M}^s$ is an isometry and $\exp: T(\mathcal{M}^s) \rightarrow \mathcal{M}^s$ is the exponential map defined by the Riemannian structure on \mathcal{M}^s (exp defined in a neighborhood of the zero section), then for any $V \in T(\mathcal{M}^s)$ if $\exp V$ is defined, $\exp(Tf(V))$ is defined and $f \exp(V) = \exp(Tf(V))$.

Before we prove this proposition we examine the definition of \exp . We use the construction in 13, pp. 109-111.

Definition 132: Let $\pi: T(\mathcal{M}^s) \rightarrow \mathcal{M}^s$, $\pi': T^*(\mathcal{M}^s) \rightarrow \mathcal{M}^s$ and $\pi^*: T(T^*(\mathcal{M}^s)) \rightarrow T^*(\mathcal{M}^s)$ be the bundle projections. The canonical one-form ω on $T^*(\mathcal{M}^s)$ is defined by $\omega(V) = \pi^*(V)(T\pi'(V))$, $V \in T(T^*(\mathcal{M}^s))$. The Riemannian structure μ (which will be noted (\cdot, \cdot)) on \mathcal{M}^s gives a bundle isomorphism $\Gamma: T(\mathcal{M}^s) \rightarrow T^*(\mathcal{M}^s)$. Let Γ^* be the map induced by Γ from forms on $T^*(\mathcal{M}^s)$ to forms on $T(\mathcal{M}^s)$. Define the two-form Ω on $T(\mathcal{M}^s)$ by $\Omega = \Gamma^*(d\omega)$.

If $f: \mathcal{M}^S \rightarrow \mathcal{M}^S$, $Tf: T(\mathcal{M}^S) \rightarrow T(\mathcal{M}^S)$.

Let $(Tf)^*$ be the induced map of forms on $T(\mathcal{M}^S)$ to forms on $T(\mathcal{M}^S)$.

Lemma 133: If f is an isometry $(Tf)^*(\Omega) = \Omega$.

Proof: Pick $X, Y \in T_V(T(\mathcal{M}^S))$ and extend them locally to vector fields which also will be called X and Y .

$$(Tf)^*(\Omega)(X, Y) = \Omega(TTf(X), TTf(Y))$$

$$= d\omega(T \lrcorner \circ TTf(X), T \lrcorner \circ TTf(Y))$$

$$(1) \qquad (2)$$

$$= (T \lrcorner \circ TTf(X))(\omega(T \lrcorner \circ TTf(Y))) - (T \lrcorner \circ TTf(Y))(\omega(T \lrcorner \circ TTf(X)))$$

$$(3) \\ - \omega(T \lrcorner \circ TTf([X, Y]))$$

$$(1) = (T \lrcorner \circ TTf(X))((Tf(V), T\pi' \circ T \lrcorner \circ TTf(Y)))$$

$$= (T \lrcorner \circ TTf(X))((Tf(V), Tf \circ T\pi(Y)))$$

$$= X((V, T\pi(Y)))$$

$$\text{Similarly } (2) = Y((V, T\pi(X)))$$

$$(3) = (Tf(V), T\pi' \circ T \lrcorner \circ TTf([X, Y]))$$

$$= (V, T\pi([X, Y])) .$$

$$\Omega(X, Y) = X((V, T\pi(Y))) - Y((V, T\pi(X))) - (V, T\pi([X, Y]))$$

by the same reasoning. q.e.d.

Let $K: T(\mathcal{M}^S) \rightarrow \mathbb{R}$ by $K(V) = \frac{1}{2}(V, V)$. dK is a one form on $T(\mathcal{M}^S)$.

Lemma 134: There exists a unique vector field Z on $T(\mathcal{M}^S)$ such that for any vector field X on $T(\mathcal{M}^S)$, $\Omega(Z, X) = -dK(X)$.

Proof: See 13, pp. 109-110.

Lemma 135: If f is an isometry of \mathcal{M}^S , $Tf(Z) = Z$.

Proof: $(Tf)^*(K) = K$, so $(Tf)^*(-dK) = -dK$.

By lemma 133, $(Tf)^*(\Omega) = \Omega$, so for any vector field X on $T(\mathcal{M}^S)$,

$$\begin{aligned} \Omega(Tf(Z), X) &= (Tf^{-1})^* \Omega(Tf(Z), X) \quad (f^{-1} \text{ also an isometry}) \\ &= \Omega(Z, Tf^{-1}(X)) \\ &= -dK(Tf^{-1}(X)) \\ &= (Tf^{-1})^*(-dK)(X) \\ &= (-dK)(X) . \end{aligned}$$

Hence $Tf(Z) = Z$ by the above lemma.

Definition 136: Let $V \in T(\mathcal{M}^S)$. Let β_V be the integral curve of the vector field Z such that $\beta_V(0) = V$.

If $\beta_V(1)$ is defined, $\exp(V)$ is defined and $\exp(V) = \pi \circ \beta_V(1)$. Z is called the exponential spray.

Proposition 137: \exp is defined on a neighborhood Θ of the zero section of $T(\mathcal{M}^S)$ and if $\text{Exp}: \Theta \rightarrow \mathcal{M}^S \times \mathcal{M}^S$ by $\text{Exp}(V) = (\pi(V), \exp(V))$, Exp is a local diffeomorphism near any zero tangent vector in $T(\mathcal{M}^S)$.

Proof: See 13, pp. 72-73.

Proof of Proposition 131: It is clear from the definition of \exp that we need only show the following: β_V and $\beta_{\text{Tf}(V)}$ have the same domain of definition and $\text{Tf} \circ \beta_V(t) = \beta_{\text{Tf}(V)}(t)$ for all t in the domain. To show this we need only check that $\text{Tf} \circ \beta_V$ is an integral curve of Z , and $\text{Tf} \circ \beta_V(0) = \text{Tf}(V)$. The latter statement is obvious. Also

$$\left. \frac{d}{dt}(\text{Tf} \circ \beta_V(t)) \right|_{t_0} = \text{Tf} \left(\left. \frac{d}{dt}(\beta_V(t)) \right|_{t_0} \right) = \text{Tf}(Z_{\beta_V(t)}) \quad \text{and} \\ \text{Tf}(Z_{\beta_V(t)}) = Z_{\text{Tf} \circ \beta_V(t)} \quad \text{by lemma 135.}$$

The proposition follows.

As a final preparatory step to proving the slice theorem, we look locally at the normal bundle ν induced by $\phi: \mathcal{D}^{s+1}/I_\gamma \rightarrow \mathcal{M}^S$.

Definition 138: Let X, Y be two manifolds, $f: X \rightarrow Y$

an immersion and $\pi: T(Y) \rightarrow Y$ be the tangent bundle.
 Then $f^*(T(Y)) = \{(x, v) \in X \times T(Y) \mid f(x) = \pi(v)\}$
 is a bundle over X with fibre $f^*(T(Y))_X \cong T_{f(x)}(Y)$.
 It is called the pull back of $T(Y)$ by f , (see 13, pp. 38-39).
 Clearly if f is injective, there is a natural map
 $\pi(f): f^*(T(Y)) \rightarrow T(Y)$ by $(x, v) \rightarrow v$, which is also
 injective.

ϕ is an injective immersion so we can find U to
 be a neighborhood of the identity coset in $\mathcal{D}^{s+1}/I_\gamma$ such
 that $\phi \upharpoonright U$ is an embedding or $\phi(U)$ is a submanifold
 of \mathcal{M}^s .

Definition 139: Let $\nu(U) = \{V \in T(\mathcal{M}^s) \mid \pi(V) = \phi(u) \in \phi(U)$
 and V is normal to the subspace $T_u\phi(T_u(\mathcal{D}^{s+1}/I_\gamma))\}$. This
 is called the normal bundle of $\phi(U)$ in \mathcal{M}^s .

Proposition 140: $\nu(U)$ is a vector bundle and $\exp \upharpoonright \nu(U)$
 is a diffeomorphism from a neighborhood of the zero section
 of $\nu(U)$ to a neighborhood of $\phi(U)$ in \mathcal{M}^s .

Proof: See 13, pp. 45-46, 73-74, and 104-105.

VIII. The Slice Theorem.

Theorem 141: At each $\gamma \in \mathcal{M}$, there exists a submanifold S of \mathcal{M}^S such that:

- 1) For all $\eta \in I_\gamma$, $\eta^*(S) = S$.
- 2) There is a neighborhood V of I_γ in $\mathcal{D}^{s+1}/I_\gamma$ such that if $\eta \in V$ and $\eta(S) \cap S \neq \emptyset$, then $\eta \in I_\gamma$.
- 3) If $\chi: U \rightarrow \mathcal{D}^{s+1}$ is a local cross section at the identity for $\pi: \mathcal{D}^{s+1} \rightarrow \mathcal{D}^{s+1}/I_\gamma$, and $F: U \times S \rightarrow \mathcal{M}^S$ is defined by $F(u, s) = (\chi(u))^*(s)$, then F is a homeomorphism onto a neighborhood of γ in \mathcal{M}^S .

Proof: Pick U a neighborhood of the identity coset, $\pi(\text{Id})$ and $\varepsilon > 0$, so that $\phi(U)$ is a submanifold of \mathcal{M}^S , and so that $\exp \upharpoonright \nu(U)$ is a diffeomorphism when restricted to the set of vectors of length $< \varepsilon$. Let \tilde{S} be an open ε -ball about zero in $\nu_{\pi(\text{Id})}(U)$, so that \tilde{S} is included in the set on which $\exp \upharpoonright \nu(U)$ is a diffeomorphism (see preceding proposition). Let $S = \exp(\tilde{S})$.

Since $\exp \upharpoonright \tilde{S}$ is an embedding, S is a submanifold.

Proof of 1): Pick $s \in S$ and let $s = \exp(t)$.

$\eta^*(s) = \eta^* \circ \exp(t) = \exp \circ T(\eta^*)(t)$ by proposition 131.

Since η^* is an isometry $\|T(\eta^*)(t)\| = \|t\| < \varepsilon$. Also, since $\eta \in I_\gamma$ and $\eta^*(\phi(\mathfrak{D}^{s+1}/I_\gamma)) = \phi(\mathfrak{D}^{s+1}/I_\gamma)$,

$$T_\gamma(\eta^*)(T_{\pi(\text{Id})}\phi(T_{\pi(\text{Id})}(\mathfrak{D}^{s+1}/I_\gamma))) = T_{\pi(\text{Id})}\phi(T_{\pi(\text{Id})}(\mathfrak{D}^{s+1}/I_\gamma)).$$

Therefore, $T\eta^*(t) \in \tilde{S}$, so $\eta^*(s) = \exp \circ T(\eta^*)(t) \in S$.

Proof of 2): Let $V = \pi^{-1}(U)$. Pick $\eta \in V$ and assume $\eta^*(s) = \eta^*(s')$. Let $s = \exp t$, $s' = \exp t'$. Then $\eta^* \exp(t) = \exp T(\eta^*)(t) = \exp t'$, $T\eta^*(t) \in \mathfrak{v}$ is a vector over $\eta^*(\gamma)$, an element of $\phi(U)$, and therefore, as in 1), $T\eta^*(t) \in \mathfrak{v}(U)$ and $\|T\eta^*(t)\| < \varepsilon$. Therefore $T\eta^*(t) = t'$ (exp being injective), so $T\eta^*(t)$ is a vector over γ . Hence $\eta^*(\gamma) = \gamma$ so $\eta \in I_\gamma$.

Proof of 3): Let $B = \{V \in \mathfrak{v}(U), \|V\| < \varepsilon\}$ and let $p: \mathfrak{v}(U) \rightarrow U$ be the bundle projection map. We shall show that F is a homeomorphism onto $\exp(B)$. Let $K: U \times \tilde{S} \rightarrow B$ be defined by $K(u, t) = T(\chi(u)^*)(t)$, K is clearly bijective. $F(u, s) = \exp \circ K(u, \exp^{-1}(s))$. Therefore, since \exp is a diffeomorphism on B , F is bijective. F is continuous since $F(u, s) = A(\chi(u), s)$ (where $A: \mathfrak{D}^{s+1} \times \mathfrak{M}^s \rightarrow \mathfrak{M}^s$ is the right action), and χ and A are continuous. Also if $W \in \exp B$, $F^{-1}(W) = (p \circ \exp^{-1}(W), A(\chi \circ \phi^{-1} \circ p \circ \exp^{-1}(W), W))$. Therefore, F^{-1} is continuous, so F is a homeomorphism. q.e.d.

IX. The Smooth Situation.

Since $A: \mathcal{D}^{s+1} \times \mathcal{M}^s \rightarrow \mathcal{M}^s$ is continuous for all large s , we know that $A: \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$ is continuous also. In particular, $\psi_\gamma: \mathcal{D} \rightarrow \mathcal{M}$ is continuous, and if \mathcal{D}/I_γ is given the quotient topology, $\phi_\gamma: \mathcal{D}/I_\gamma \rightarrow \mathcal{M}$ is a continuous injection.

Proposition 142: $\phi_\gamma: \mathcal{D}/I_\gamma \rightarrow \mathcal{M}$ is a homeomorphism onto a closed subset of \mathcal{M} . The proof will be a sequence of lemmas.

Lemma 143: Given any sequence $\{\eta_m\} \subseteq \mathcal{D}$ and finite set $\{p_i\}$ of points in M . There is a subsequence $\{\zeta_n\}$ of $\{\eta_m\}$ such that for all i , there is some q_i in M , such that $\zeta_n(p_i) \rightarrow q_i$.

Proof: For a single p_i , $\eta_m(p_i)$ is a sequence in a compact set. Hence it has a convergent subsequence $\zeta_n(p_i)$, so $\{\zeta_n\}$ satisfies the lemma. If $\{p_i\} = \{p_1 \dots p_{j+1}\}$, assume there is a sequence ζ_n so that $\zeta_n(p_i) \rightarrow q_i$ for $i \leq j$. Then refine ζ_n so that $\zeta_n(p_{j+1})$ converges as well. This refinement satisfies the lemma.

Lemma 144: Fix $\gamma, \gamma' \in \mathcal{M}$ and let $\{\eta_m\}$ be a sequence in \mathcal{D} such that $\eta_m^*(\gamma) \rightarrow \gamma'$. Then for any finite

collection of vectors $\{V_i\}$ in $T(M)$, there exist vectors W_i and a subsequence $\{\zeta_n\}$, such that for all i , $T\zeta_n(V_i) \rightarrow W_i$. Also, if $V_i \neq 0$, $W_i \neq 0$.

Proof: Let K be the maximum of the lengths of the vectors V_i with respect to γ' .

$K = \max\{(\gamma'(V_i, V_i))^{1/2}\}$. Since $\eta_m^*(\gamma) \rightarrow \gamma'$, for each i , $\gamma(T\eta_m V_i, T\eta_m V_i) \rightarrow \gamma'(V_i, V_i)$. Therefore, for sufficiently large m , $\gamma'(T\eta_m V_i, T\eta_m V_i)^{1/2} \leq 2K$. Let

$S_{2K}(M) = \{V \in T(M) \mid (\gamma'(V, V))^{1/2} \leq 2K\}$. $S_{2K}(M)$ is compact, and for large m , $T\eta_m(V_i) \in S_{2K}(M)$. Now we can construct a subsequence as in the previous lemma. If $V_i \neq 0$ and $W_i = 0$ then $\zeta_k^*(\gamma)(V_i, V_i) \rightarrow 0$ so $\gamma'(V_i, V_i) = 0$ which is impossible. q.e.d.

Lemma 145: Let $e: T(M) \rightarrow M$ be the exponential map of γ and $e_m: T(M) \rightarrow M$ be that of $\eta_m^*(\gamma)$. Then $\eta_m e = e_m T\eta_m$.

Proof: We have proved this in the case that η_m is an isometry (see proposition 131). We use the same proof

here: Let $K, \Omega,$ and Z be the function, two-form, and spray defined by γ on $T(M)$ (see lemmas 133-135), and let K^m, Ω^m and Z^m be the corresponding objects for $\eta_m^*(\gamma)$. Then $K^m = (T\eta_m)^*(K)$, $\Omega^m = (T\eta_m)^*(\Omega)$ and $Z^m = T\eta_m Z$.

Therefore if $\beta_V(t)$ is an integral curve of Z through V , $T\eta_m \circ \beta_V(t)$ is an integral curve of $TT\eta_m Z$ through $T\eta_m(V)$. The proposition follows.

Proposition 146: Let $e': T(M) \rightarrow M$ be the exponential map of γ' . Then $e_m \rightarrow e'$ uniformly in all derivatives on compact subsets of $T(M)$.

Proof: Let K', Ω' and Z' be the function, two form and spray defined by γ' on $T(M)$. Then, since $\eta_m^*(\gamma) \rightarrow \gamma'$, $K^m \rightarrow L'$, $\Omega^m \rightarrow \Omega'$ and $Z^m \rightarrow Z'$: all convergences being uniform in all derivatives on compact subsets.

If $V \in T(M)$, $e(V) = \pi \circ \beta(1, V)$, where $\pi: T(M) \rightarrow M$ and $\beta(1, V) = V + \int_0^1 Z_{\beta(t, V)} dt$, the integral curve of Z . Convergence for such a case is proven in lemma 94.

Proof of Proposition 142: Since M is compact, there exists a number $\varepsilon > 0$, such that any ball in M (with respect to γ) of radius less than ε lies entirely in some normal coordinate neighborhood of M . Also for γ' there exists and $\varepsilon' > 0$ with the same property.

Let $K = \max_{V \in T(M)} \{\gamma'(V, V)/\gamma(V, V)\}$. Pick a finite set $\{p_i\}$ of M such that for each i , U_i is a normal

(with respect to γ) coordinate neighborhood centered at p_i , of radius less than $\delta = K^{-1/2} \min(\epsilon, \epsilon')$ and $\bigcup_i U_i = M$. Pick $\{V_i^j\}$ such that for each fixed i , $\{V_i^j\}_j$ is an orthonormal basis (with respect to γ) of $T_{p_i}(M)$.

Find $\{W_i^j\} \subseteq T(M)$ and a subsequence $\{\zeta_k\}$ such that $\zeta_k(p_i) \rightarrow q_i$ and $T_{\zeta_k}(V_i^j) \rightarrow W_i^j$ for all i, j . $\{W_i^j\}_j$ is a basis of $T_{q_i}(M)$ by lemma 144.

Claim: There exists $\zeta \in \mathcal{D}$, such that $\zeta_k \rightarrow \zeta$ in \mathcal{D} .

Consider a fixed U_i about p_i . If $q \in U_i$ $q = e(\sum_j a_i^j V_i^j)$ where $\sum_j (a_i^j)^2 < \delta$. Therefore, $\zeta_k(q) = \zeta_k e(\sum_j a_i^j V_i^j) = e \circ T_{\zeta_k}(\sum_j a_i^j V_i^j)$, $T_{\zeta_k}(V_i^j) \rightarrow W_i^j$ so $\{T_{\zeta_k}(V_i^j)\}_k$ is a bounded set. Also $e_k \rightarrow e'$ uniformly on bounded sets, so $\zeta_k(q) \rightarrow e'(\sum_j a_i^j W_i^j)$. Let $\zeta(q) = e'(\sum_j a_i^j W_i^j)$.

Then $\zeta_k \rightarrow \zeta$ on U_i . Extend ζ to be a map on M by defining it on each U_i . Since $\{\zeta_k\}$ are maps on M and $\zeta_k(q) \rightarrow \zeta(q)$, it is clear that for $q \in U_i \cap U_{i'}$. The two constructions of $\zeta(q)$ (from U_i and $U_{i'}$) must coincide. Therefore ζ is well defined on M and since $\zeta_k \rightarrow \zeta$ on each U_i , $\zeta_k \rightarrow \zeta$ on M . On U_i , $\zeta = e' \circ L \circ e_{p_i}^{-1}$ where $e_{p_i}: T_{p_i}(M) \rightarrow M$ is $e|_{T_{p_i}(M)}$,

and $L: T_{p_1}(M) \rightarrow T_{q_1}(M)$ is the linear map defined by $L(\sum a_i j_{V_i}^j) = \sum a_i j_{W_i}^j$. Since $\gamma'(W_i^j, W_i^j) \leq K$, $L \circ e_{p_1}^{-1}(U_1)$ is contained in a neighborhood of zero of radius $\leq \varepsilon'$ (with respect to γ'). Therefore $e' \upharpoonright L \circ e_{p_1}^{-1}(U_1)$ is a diffeomorphism. Hence $\zeta \upharpoonright U_1$ is a diffeomorphism onto a neighborhood of $\zeta(p_1)$.

To show that ζ is a diffeomorphism we need only check that it is one to one and onto. Since $\zeta \upharpoonright U_1$ is a diffeomorphism, $\zeta(M)$ is open in M . But M is compact, so $\zeta(M)$ is closed in M . Hence ζ is onto.

Now we show ζ is injective. Let ρ , ρ_k and ρ' be the metrics on M induced by γ , $\zeta_k^*(\gamma)$, and γ' respectively. Let ℓ be the Lebesgue number of the covering $\{U_i\}$ with respect to ρ ; i.e., if $\rho(p, q) < \ell$, there is some U_i such that $p, q \in U_i$. We know $\zeta \upharpoonright U_i$ is injective. So we consider $p, q \in M$ where $\rho(p, q) \geq \ell$. Then $\rho_k(\zeta_k(p), \zeta_k(q)) \geq \ell$ for all k . But $\zeta_k^*(\gamma) \rightarrow \gamma'$ and $\zeta_k(p) \rightarrow \zeta(p)$, $\zeta_k(q) \rightarrow \zeta(q)$. Therefore, $\rho'(\zeta(p), \zeta(q)) \geq \ell$, so $\zeta(p) \neq \zeta(q)$. Hence ζ is a diffeomorphism. Since $\zeta_k \rightarrow \zeta$ in \mathcal{D} , $\zeta_k^*(\gamma) \rightarrow \zeta^*(\gamma)$ so $\zeta^*(\gamma) = \gamma'$. This shows that the orbits of \mathcal{D} in \mathcal{M} are closed. Now we show $\phi: \mathcal{D}/I_\gamma \rightarrow \mathcal{M}$ a homeomorphism into. Let $\eta_m^*(\gamma) \rightarrow \eta^*(\gamma)$. Then

$(\eta_m \circ \eta^{-1})^*(\gamma) \rightarrow \gamma$. If we can show that $I_\gamma \eta_m \circ \eta^{-1} \rightarrow I_\gamma$ in \mathcal{D}/I_γ , then $I_\gamma \eta_m \rightarrow I_\gamma \eta$ --- the desired conclusion. Therefore assume $\eta_m^*(\gamma) \rightarrow \gamma$ and that there is a neighborhood U of I_γ in \mathcal{D}/I_γ and a subsequence $\{\eta_{m_1}\}$ such that $\{\eta_{m_1}\} \cap U = \emptyset$. $(\eta_{m_1})^*(\gamma) \rightarrow \gamma$, so there is a subsequence $\{\xi_k\}$ of $\{\eta_{m_1}\}$ and a diffeomorphism ξ such that $\xi_k \rightarrow \xi$ and $\xi \in I_\gamma$. Therefore for large k , $\xi_k \in U$. Contradiction. Hence if $\eta_m^*(\gamma) \rightarrow \gamma$, $I_\gamma \eta_m \rightarrow I_\gamma$ in \mathcal{D}/I_γ . The proposition follows.

Using the fact that $\phi: \mathcal{D}/I_\gamma \rightarrow \mathcal{M}$ is a homeomorphism into, and using the invariant metric on \mathcal{M}^S , one can prove the following:

Theorem 147: If for $\gamma \in \mathcal{M}$, I_γ is the trivial group, then there exists a neighborhood V of γ , such that for all $\gamma' \in V$, $I_{\gamma'}$ is trivial. Also if $\dim M > 1$, for any $\gamma \in \mathcal{M}$ there is a γ' in any neighborhood of γ , such that $I_{\gamma'} = \{\text{Id}\}$. That is, the set $\mathcal{G} = \{\gamma \in \mathcal{M} | I_\gamma = \{\text{Id}\}\}$ is open and dense in \mathcal{M} . We call it the set of generic metrics.

Preparatory to the proof of this theorem, we construct an invariant metric on \mathcal{M} . For $t \geq s$, let ρ_t be the metric on \mathcal{M}^t induced by its Riemannian structure as defined in section IV.

Let $\rho: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ by

$$\rho(\gamma, \gamma') = \sum_{t \geq s} \frac{1}{2^t} \left(\frac{\rho_t(\gamma, \gamma')}{1 + \rho_t(\gamma, \gamma')} \right).$$

Proposition 148: ρ is a metric on \mathcal{M} which is invariant under the action of \mathcal{D} and which induces the given topology on \mathcal{M} .

Proof: That ρ is an invariant metric is immediate from the fact that each ρ_t is. Also, if $\gamma_n \rightarrow \gamma$ in \mathcal{M} $\gamma_n \rightarrow \gamma$ in \mathcal{M}^t for all t , so $\rho_t(\gamma_n, \gamma) \rightarrow 0$, and therefore, $\rho(\gamma_n, \gamma) \rightarrow 0$. Conversely if $\rho(\gamma_n, \gamma) \rightarrow 0$, then for each t , $\rho_t(\gamma_n, \gamma) \rightarrow 0$. But $\rho_s(\gamma_n, \gamma) \rightarrow 0$ implies that $\gamma_n \rightarrow \gamma$ in the C^1 sense and generally $\rho_t(\gamma_n, \gamma) \rightarrow 0$ implies that $\gamma_n \rightarrow \gamma$ in the C^{t-s} sense. Therefore, $\gamma_n \rightarrow \gamma$, C^k for all k , or $\gamma_n \rightarrow \gamma$, C^∞ . This proves the proposition.

Proof of Theorem 147: If $\rho(\gamma, \gamma') < \varepsilon$ and $\eta \in I_{\gamma'}$, then $\rho(\eta^*(\gamma), \gamma') < \varepsilon$ so $\rho(\eta^*(\gamma), \gamma) < 2\varepsilon$. Hence by choosing V a small neighborhood we can insure that

$$\max_{\substack{\eta \in I_{\gamma'} \\ \gamma' \in V}} \{\rho(\eta^*(\gamma), \gamma)\} \text{ is small. Since } \phi_\gamma: \mathcal{D} \rightarrow \mathcal{M} \text{ is a}$$

homeomorphism into, this means that we can require that

the set $\bigcup_{\substack{\eta \in I_{\gamma'} \\ \gamma' \in V}} \{\eta\}$ is within any fixed neighborhood of Id

in \mathcal{D} . Hence to prove the openness of \mathcal{S} we need only find a neighborhood V of γ , and a neighborhood W of Id in \mathcal{D} such that if $\gamma' \in V$, $I_{\gamma'} \notin W$.

Fix a point $p \in M$ and let U be a convex normal coordinate neighborhood of p with radius $3a$ (with respect to γ).

That is, $e \upharpoonright T_p(M)$ is a diffeomorphism from the open ball about zero of radius $3a$ onto U , and for any points q_1, q_2 in U , there is a unique geodesic from q_1 to q_2 of length less than $6a$. Also, if q_1 and q_2 are within some distance ε of p , so is the entire geodesic segment between them, so this geodesic lies entirely inside U , (for existence of U see 3, pp. 246-248).

Pick V a neighborhood of γ such that for all $\gamma' \in V$, U contains a convex neighborhood of p with radius bigger than $2a$ with respect to γ' , and for any $\eta \in I_{\gamma'}$ and $q \in M$, $d(q, \eta(q)) < a$ and $d'(q, \eta(q)) < a$, where d and d' are the metrics on M induced by γ and γ' respectively.

Let O_n be the orthogonal group for \mathbb{R}^n where $\dim M = n$. Given $\gamma' \in V$, we shall define a map $f: I_{\gamma'} \rightarrow O_n$.

We first define a function $\chi : U \rightarrow F(U)$ where $F(U)$ is the bundle of orthogonal frames over U with respect to γ' . Fix an orthogonal frame F_p over p and define $\chi(q)$ to be the parallel translate of F_p to q along the unique shortest geodesic from p to q . Now pick $\eta \in I_{\gamma'}$ and define $f(\eta)$ by the equation $f(\eta)\chi(\eta(p)) = F\eta F_p$ where $F\eta$ is the map induced by η on the bundle of frames over M .

Claim: If $\eta \in I_{\gamma'}$ and $\eta \neq \text{Id}$, then there is a power η^i of η and a point x on the $(n-1)$ sphere such that the angle between $f(\eta^i)(x)$ and x is greater than or equal to $\pi/2$.

Case I: Assume $\eta(p) = p$. Then $f(\eta) = \text{Id} \in O_n$ only if $T\eta = \text{Id}$. But it is well known that if $T_p\eta = \text{Id} : T_p(M) \rightarrow T_p(M)$ and η is an isometry, then $\eta = \text{Id}$. (See 12, pp. 131, 138.) Clearly, $f(\eta^i) = f(\eta)^i$ in this case, so our claim is an elementary property of O_n .

Case II: $\eta(p) \neq p$.

Assume the claim false and let ζ be some power of η .

Let $b = d'(\zeta(p), p)$ and let g_1, g_2 be the geodesics from p to $\zeta(p)$ and $\zeta(p)$ to $\eta\zeta(p)$ respectively.

Let V_1 be the tangent to g_1 at $\zeta(p)$ and V_2 the tangent to g_2 at $\zeta(p)$. Then the angle between V_1 and V_2 is less than $\pi/2$. Let S_b be the sphere around p of radius b with respect to d' . $\zeta(p) \in S_b$ and by the Gauss lemma, V_1 is perpendicular to S_b . Therefore V_2 points outside of S_b , so part of the geodesic g_2 is farther from p than the distance b . Hence $d'(p, \eta\zeta(p)) > b$, because $\eta\zeta(p)$ and $\zeta(p)$ are in a convex neighborhood of p .

Using induction we can state the following:

The sequence $\eta(p), \eta^2(p), \eta^3(p), \dots$ gets continually farther from p .

We recall a trivial property of compact groups:

Lemma 149: If G is a compact topological group, given an element η of G and a neighborhood W of the identity, there exists a power η^i of η such that $\eta^i \in W$.

Proof: Assume the lemma false, and let V be a neighborhood of the identity such that $VV^{-1} \subseteq W$. Then we can check that the set $\{\eta^i V\}$ is a disjoint collection. The set $\{\eta^i\}$ is closed and therefore compact. But the cover $\{\eta^i V\}$ of $\{\eta^i\}$ has no finite subcover. The lemma follows.

$I_{\gamma'}$ is a compact group and by the above, $\eta^i(p)$ does not approach p , so η^i does not approach Id . This is a contradiction; it establishes our claim.

We shall now show that by shrinking V slightly, we can insure that if $\gamma' \in V$, $\eta \in I_{\gamma'}$, there is no x on the $(n-1)$ sphere such that the angle between x and $f(\eta)(x)$ is bigger than or equal to $\pi/2$. This, of course, will conclude our proof of the openness of \mathcal{B} .

Assume that a is small enough such that for all $q \in M$, $e|_{T_q(M)}$ is a diffeomorphism when restricted to a $3a$ -ball about zero, and restrict V so that for any $\gamma' \in V$, the above is true for a $2a$ -ball using e' , the exponential associated to γ' .

Then, for any $q \in U$, there is a $c > 0$ such that if $v', v \in T_q(M)$ and $\|v\| = \|v'\| = a$ and the angle between v and v' is not less than $\pi/2$, (i.e., $\gamma(v, v') \leq 0$) then $d(e(v), e(v')) > 4c$. We further restrict V , so that for $\gamma' \in V$ and corresponding v, v' $d'(e'(v), e'(v')) > 3c$.

Now find h , $a > h > 0$, such that if $d(p, q) < h$ and $v_p \in T_p(M)$ such that $\|v\| = a$, then if $q \in U$ and v_q is the parallel translate of v_p to q along the shortest geodesic through p and q , then

$d(e(v_p), e(v_q)) < c$. Restrict V so that for all $\gamma' \in V$, and corresponding v_p, v_q , $d'(e'(v_p), e'(v_q)) < 2c$.

Finally restrict V so that for any $\gamma' \in V$, $\eta \in I_{\gamma'}$, and $q \in M$, $d'(q, \eta(q)) < \min(c, h)$.

Now pick $\gamma' \in V$, $v_p \in T_p(M)$ such that the γ' length of v_p is a . Pick $\eta \in I_{\gamma'}$ and let $q = \eta(p)$, let v_q be the translate of v_p to q . Then

$$d'(e'(T\eta v_p), e'(v_q)) \leq d'(e'(v_p), e'(T\eta(v_p))) + d'(e'(v_q), e'(v_p)) .$$

But $e'(T\eta v_p) = \eta e'(v_p)$ so the first term on the right is less than c . Also the second term is less than $2c$ since v_q is a parallel translate of v_p . Therefore, $d'(e'(T\eta v_p), e'(v_q)) < 3c$ so the angle between $T\eta v_p$ and v_q is less than $\pi/2$. This shows that \mathcal{L} is open.

Now to show that \mathcal{L} is dense we look at the curvature tensor and perturb it slightly. Fix $\gamma \in \mathcal{M}$ and let $T_-(M)$ be the non-zero vectors in $T(M)$. Let $f_\gamma: T_-(M) \rightarrow \mathbb{R}$ by $f_\gamma(X) = \text{Ric}(X, X) / \langle X, X \rangle$, where $\text{Ric}(X, X)$ denotes the Ricci curvature, (see 11, p. 95). f is a smooth function on $T_-(M)$ and if $\{x^i\}$ are a set of normal coordinates in a neighborhood U of p , such that the components g_{ij} of γ are the identity matrix at p , then

$$f_{\gamma}(X_n) = \frac{1}{2} \sum_i \left(\frac{\partial^2 g_{ii}}{\partial x^n^2} - 2 \frac{\partial^2 g_{ni}}{\partial x^n \partial x^i} + \frac{\partial^2 g_{nn}}{\partial x^i^2} \right) .$$

(This is a direct calculation using the standard formulae for curvature and affine connections, see 11, p. 63).

Let $S(M)$ be the set of unit vectors in M (with respect to γ). f_{γ} has a maximum on $S(M)$ and since for $\lambda \in \mathbb{R} - \{0\}$, $f_{\gamma}(\lambda X) = f_{\gamma}(X)$, this maximum is the maximum of f_{γ} on $T_-(M)$. We shall find γ' arbitrarily near γ so that $f_{\gamma'}$ has a greater maximum.

Take a set of smooth functions $\{\varepsilon_{ij}\}$ on U (symmetric in i and j) whose support is in U , such that for all i, j, k , $\varepsilon_{ij}(p) = \frac{\partial \varepsilon_{ij}}{\partial x^k} \Big|_p = 0$. Let γ' be the metric on M defined by: $\gamma' = \gamma$ off of U and $\gamma' \sim g_{ij} + \varepsilon_{ij}$ over U . Then $\{x^i\}$ are normal coordinates at p for γ' also, and at p , $f_{\gamma'}(X_n) = f_{\gamma}(X_n)$

$$+ \frac{1}{2} \sum_i \left(\frac{\partial^2 \varepsilon_{ii}}{\partial x^n^2} - 2 \frac{\partial^2 \varepsilon_{ni}}{\partial x^n \partial x^i} + \frac{\partial^2 \varepsilon_{nn}}{\partial x^i^2} \right) .$$

It is clear that if $n > 1$, there are ε_{ij} arbitrarily close to zero in $C_0^{\infty}(U)$, such that $f_{\gamma'}(X_n) > f_{\gamma}(X_n)$. (If $n = 1$, $f_{\gamma} = f_{\gamma'} = 0$.)

This construction tells us the following:

1) At any point p of M and unit vector $X \in T_p(M)$, and for any neighborhood U of p , we can find a Riemannian metric γ' arbitrarily near to γ such that $\gamma' = \gamma$ off of U and $f_{\gamma'}(X) > f_{\gamma}(X)$.

Also, the following is immediate from the definition of f_{γ} :

2) For any $\gamma \in \mathcal{M}$, if $\eta \in I_{\gamma}$, $f_{\gamma}(T\eta X) = f_{\gamma}(X)$ for any $X \in T_-(M)$.

Furthermore, from the proof of the openness of \mathcal{G} we know:

3) For any $p \in M$, there are neighborhoods U of p in M , and V of γ in \mathcal{M} and there is a $\delta > 0$ such that for any $\gamma' \in V$, $I_{\gamma'} = \{\text{Id}\}$ or for some $\eta \in I_{\gamma'}$, there exists $q \in U$ such that $d(q, \eta(q)) > \delta$ where d is the metric on M induced by γ .

Using the above three statements, it is easy to show the denseness of \mathcal{G} . We select $p \in M$ so that f_{γ} is maximal at $X \in T_p(M)$. Let U be a convex neighborhood of p of radius $3a$ for which 3) holds. Pick $b_0 < a$ and let $B_p^{b_0}$ be the neighborhood of p of radius b_0 . Perturb γ on $B_p^{b_0}$ as in 1) to get γ_0 so that $f_{\gamma_0}(X) > f_{\gamma}(X)$. Then for any $Y \in T_-(M)$, such that

$Y \notin T_-(B_p^{b_0})$, $f_{\gamma_0}(X) > f_{\gamma_0}(Y)$. Therefore by 2) , if $\eta \in I_{\gamma_0}$, $T\eta(X) \in T_-(B_p^{b_0})$ so $\eta(p) \in B_p^{b_0}$.

Let $S_{2a} = \{g \in M | d_0(p, q) = 2a\}$ where d_0 is the metric on M defined by γ_0 . Then if $q \in S_{2a}$ and $\eta \in I_{\gamma_0}$, $2a - b_0 \leq d_0(p, \eta(q)) \leq 2a + b_0$.

Let $A_1 = \{q \in M , 2a - b_0 \leq d_0(p, q) \leq 2a + b_0\}$ and pick $p_1 \in A_1$ such that there is $X_1 \in T_{p_1}(M)$ such that

$$\max_{Y \in T_-(A_1)} \{f_{\gamma_0}(Y)\} = f_{\gamma_0}(X_1) .$$

Now perturb γ_0 on a b_1 -ball $B_{p_1}^{b_1}$ about p_1 to get γ_1 such that

$$f_{\gamma_0}(X_1) < f_{\gamma_1}(X_1) < f_{\gamma_1}(X) = f_{\gamma_0}(X) . \text{ Then if } \eta \in I_{\gamma_1} ,$$

$\eta(p) \in B_p^{b_0}$ and $\eta(p_1) \in B_{p_1}^{b_1}$. Let d_1 be the metric

induced by γ_1 , and let $A_2 = \{q \in M | 2a - b_0 \leq d_1(p, q) \leq 2a + b_0$

and $a - b_1 \leq d_1(p, q) \leq a + b_1\}$. Pick $p_2 \in A_2$ and

$X_2 \in T_{p_2}(M)$ so as to maximize f_{γ_1} .

Continue inductively to get $b_0 \cdots b_n$, $A_1 \cdots A_n$, $p_1 \cdots p_n$ and $\gamma_1 \cdots \gamma_n$. We claim that $I_{\gamma_n} = \{\text{Id}\}$.

Let $E_n: T_p(M) \rightarrow M \times M$ be the exponential map of γ_n ,

and let $Y_i = E_n^{-1}(p, p_i)$, $i=1, \dots, n$.

Case I: $\{Y_i\}$ are independent vectors in $T_p(M)$. Thus for any $q \in U$, $q = e_n(\sum_i y_i^i Y_i)$ so if $\eta \in I_{\gamma_n}$, $\eta(q) = e_n(\sum_i y_i^i T_\eta(Y_i))$. But by making $b_0 \cdots b_n$ small we can insure that $T_\eta Y_i$ is arbitrarily close to Y_i , (since the inequalities

$$(*) \quad f_{\gamma_n}(X_n) < f_{\gamma_n}(X_{n-1}) \cdots f_{\gamma_n}(X_1) < f_{\gamma_n}(X)$$

and

$$(**) \quad f_{\gamma_n}(X_i) > \max_{Y \in T(A_i - B_{p_i}^{b_i})} \{f_{\gamma_n}(Y)\}$$

imply that $\eta(p) \in B_p^{b_0}$ and $\eta(p_i) \in B_{p_i}^{b_i}$, and hence that $\eta(q)$ is arbitrarily close to q . By 3), this means $\eta = \text{Id}$.

Case II: If $\{Y_i\}$ are not independent, move the p_i slightly so that they are. Clearly this can be done without annulling the inequalities, (*) and (**).

Therefore, by the argument of case I, $I_{\gamma_n} = \{\text{Id}\}$. Now it is clear that \mathcal{S} is dense in \mathcal{M} , so the proof of theorem 147 is complete.

X. Further Research.

The present work gives rise to several avenues of additional research. Perhaps the broadest one is that of further applications of H^S maps from one manifold to another. This is the first instance known to the author of the use of H^S maps, but they should have many other possibilities. They are preferable to C^k functions in that the manifold of H^S maps is based on a Hilbert space rather than a Banach space, and in that they are a more natural tool when discussing the range and Kernel of a differential operator.

One thing that might be done using them, is to study the orbits of \mathcal{D} (or \mathcal{D}^S) in spaces of other structures (e.g. conformal, complex, semi-Riemannian) on a compact manifold. Eells and Earle have done this with conformal structures in the case $\dim M = 2$, (see 8).

As to more particular results, it would be nice to show that the map $\phi: \mathcal{D}^{S+1}/I_\gamma \rightarrow \mathcal{M}^S$ is a homeomorphism onto a closed subset of \mathcal{M}^S and by this, to get the full classical slice theorem. This would tell us that the orbits of \mathcal{D}^{S+1} in \mathcal{M}^S are closed and give a first step towards investigation of the quotient space $\mathcal{M}^S/\mathcal{D}^{S+1}$.

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BIOGRAPHY

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